JASET KMUTNB
Research Article

Journal of Applied Science and Emerging Technology (JASET)
Published by Faculty of Applied Science, KMUTNB, Thailand
doi: 10.14416/JASET.KMUTNB.2024.02.004
Vol. 23, No. 2 [2024]: e257034

Development of average run length formulas for EWMA control chart under the AR(1) with quadratic trend model for detecting and monitoring process variability

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ABSTRACT

This paper proposes an explicit formula for calculating the average run length (ARL) of an exponentially weighted moving average (EWMA) control chart for monitoring process mean shifts. The study focuses on the AR(1) model with quadratic trends, because this model combines two essential components in time series analysis including the autocorrelation and long-term trend analysis. It is useful for producing accurate and insightful data predictions. Additionally, it is used in various contexts such as economic forecasting, financial analysis, and environmental studies. Furthermore, the study presents a technique for estimating the ARL using the numerical integral equation (NIE) method. This enables a comparison between the results of the explicit formula and the numerical integral equation method. The two ARL solutions obtained from the explicit formula and numerical integral equation method are very similar identical with an absolute percentage difference of less than 0.001. Thereby, the explicit formula accurately corresponds to the NIE method. Additionally, the explicit formulas are more computationally efficient as they require fewer computations compared to the NIE approach

KEYWORDS: Average run length, AR(1) with quadratic trend, Explicit formulas, Numerical integral equation.

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Received: 05/06/2024; **Revised:** 26/07/2024; **Accepted:** 30/07/2024

1. INTRODUCTION

Control charts are important tools for statistical process control. In 1931, Shewhart introduced the Shewhart control chart, which is used to detect a large shift in the process mean or variance. It is especially effective when the observations have a normal distribution (Shewhart, 1931). Subsequently, Page introduced the Cumulative Sum (CUSUM) control chart. It is adept at identifying small shifts in statistical parameters and complex patterns like autocorrelation (Page, 1954). Recently, Robert proposed the exponentially weighted moving average (EWMA) control chart (Robert, 1959). The CUSUM and EWMA control charts are more effective in identifying small changes and complex patterns within processes, whereas the Shewhart control chart is effective in detecting large shifts.

Statistical quality control techniques will be used in this research to manage process variability. In general, the effectiveness of control chart research frequently depends on an initial assumption that the data has a normal distribution. However, in many real-world scenarios, data usually displays a time series trend. Therefore, for efficient process change monitoring, choosing a suitable control chart is important. Therefore, the average run length (ARL) of EWMA control charts is of interest to the researchers, and they are working on creating a precise formula and estimation technique for it.

The ARL is a critical metric for assessing control chart effectiveness in detecting shifts in process mean. It represents the average number of subgroups needed before a control chart detects an out-of-control process. ARL comprises two components: ARL_0, the average number of samples from a stable process before a false out-of-control signal, and ARL_1, the average number of samples within control limits before an out-of-control signal is triggered.

Three common methods for estimating the ARL are Monte Carlo simulations (MC), Markov Chain approach (MCA), and Integral Equation approach (IE). Roberts (1959) used the MC method to calculate the ARL for the EWMA control chart, which is useful for validating analytical results but can be time-consuming. Crowder (1987) introduced a numerical procedure for EWMA run length computation, extending to non-normal cases and one-sided EWMA control charts. Later, Lucas and Saccucci (1990) showed that the EWMA control scheme, used to monitor process mean shifts by using the MCA method. They proposed a design procedure for EWMA control chart with different parameter values that can efficiently detect small shifts in a process. The limitations of both MCA and MC methods have caused researchers to reconsider their research on the IE approach. Analytical solutions for average delay and ARL on EWMA control charts with observations of exponential distribution were proposed by Areepong and Novikov (2009). After that, Mititelu et al. (2010) used the Fredholm integral equations to derive explicit formulas for ARL in special control charts like CUSUM and EWMA, requiring fewer computations.

Control charts are commonly formulated assuming observations are independent and identically distributed. When a process displays indications of autocorrelation, it becomes essential to make use of specialized control charts. Vanbrackle and Reynolds (1997) discovered that correlation has a significant impact on the ARL and steady state ARL of EWMA and CUSUM control charts, even though independence is assumed in control chart evaluation. They provided numerical evaluations for control chart design using integral equation and MCA methods. Busaba et al. (2012) proposed the numerical integral equation (NIE) method to approximate the ARL_0 and ARL_1 in the CUSUM procedure. They used the first order autoregressive (AR(1)) model with exponential white noise, which showed an excellent agreement between the NIE method and the explicit formula. Later, Petcharat et al. (2013) used a moving average (MA) model to provide explicit formulas of ARL for EWMA and CUSUM control charts. Phanyaem (2022) used the IE and NIE methods to evaluate the ARL of the CUSUM chart on the $SARX(P,r)_L$ model. The Fredholm integral equation was employed, and NIE methods like the midpoint rule, the trapezoidal rule, Simpson's rule, and the Gaussian rule were used to approximate the ARL. Recently, Petcharat (2022) constructs the ARL for a CUSUM control chart using the Fredholm integral equation approach and Banach's Fixed Point theorem to ensure the solution's existence and uniqueness based on $SAR(P)_L$ with the trend process. Subsequently, Karoon and Areepong (2023) confirmed the efficacy of the double exponential weighted moving average (DEWMA) control chart in monitoring process quality by creating an explicit ARL formula using an autoregressive model with trend. Furthermore, Karoon et al. (2023) studied the ARL of the double exponentially weighted moving average (double EWMA) control chart for observational data following exponential white noise in a time series model with an autoregressive process. Supharakonsakun and Areepong (2023) developed a double exponentially weighted moving average (DEWMA) control chart to detect small shifts in a moving average of order q (MA(q)) process with exponential white noise. Karoon et al. (2024) show the effectiveness of the extended exponentially weighted moving average (EEWMA) control chart in detecting small shifts, particularly when the observations are autocorrelated with exponential white noise, using explicit formulas for the ARL. Recently, Sunthornwat et al. (2024) developed a specific formula for the ARL for the autoregressive process with a quadratic trend on a modified exponentially weighted moving average (EWMA) control chart. Most recently, Phanthuna et al. (2024) presented a double-modified exponentially weighted moving average (DMEWMA) chart for an autoregressive (AR) process.

In this paper, we propose the explicit formula for ARL of the EWMA control chart for an AR(1) with quadratic trend model. This paper presents a new contribution that has not been explored before. The proposed ARL of the EWMA control chart is compared with NIE approaches. The paper is structured as follows: Section 2 explains the materials used; Sections 3 and 4 describe the methods employed; Section 5 presents the proposed method's results, and Section 6 provides concluding remarks.

2. MATERIALS AND METHODS

This section explains the features of AR(1) with quadratic trend model, which is used on the EWMA control chart for monitoring process mean shifts. The last section examines the characteristics of the ARL that are essential to the assessment of control chart performance.

2.1 AR(1) with Quadratic Trend model

The AR(1) with quadratic trend model is a time series model that combines the AR(1) model with quadratic trend component. In time series analysis, it is common to assume that white noise follows a normal distribution. However, the choice of using an exponential distribution or other types of distributions for error terms depends on the characteristics of the data being studied. In this case, we are assuming that the white noise follows an exponential distribution. An exponential distribution for the white noise error term is important when the time series data or the underlying process is inherently non-negative. The model analyzes the serial correlation of time series data using AR model and quadratic trend; coefficients c, ϕ, β_1 and β_2 are estimated statistically using techniques such as maximum likelihood estimation.

An AR(1) model with quadratic trend can be represented by equation (1).

$$Y_t = c + \phi Y_{t-1} + \beta_1 t + \beta_2 t^2 + \varepsilon_t \,;\, t = 1,2,... \quad (1)$$

where c is a suitable constant, ϕ is an autoregressive coefficient, β_1 is a linear coefficient, β_2 is a quadratic coefficient, t is the times and ε_t is the exponential white noise sequence of independent and identically distributed random variables.

2.2 EWMA Control Chart

An EWMA control chart is a statistical tool used in process control to monitor the mean or variance of a process over time, an alternative to conventional control charts like the Shewhart control chart. The EWMA statistic is calculated as a weighted average of the current observation and the previous EWMA statistic. It provides more weight for current observations, making it sensitive to changes or trends in the process.

The EWMA statistic is represented by equation (2).

$$E_t = (1-\lambda)E_{t-1} + \lambda Y_t \,;\, t=1,2,... \quad (2)$$

where E_{t-1} is a previous EWMA statistic and λ is an exponential smoothing parameter with $0 < \lambda < 1$.

The upper and lower control limits of the EWMA control chart are as follows:

$$UCL = \mu + L\,\sigma\sqrt{\frac{\lambda}{2-\lambda}}, \quad LCL = \mu - L\,\sigma\sqrt{\frac{\lambda}{2-\lambda}},$$

where μ and σ are the mean and standard deviation of the process and L is a suitable control width limit.

3. INTEGRAL EQUATION METHOD

In this section, we derive analytical formulas for ARL using the Fredholm Integral Equation of the second kind. The lower and upper control limits are determined as zero and b respectively. The function $H(u)$ is defined as the ARL of EWMA control chart for the AR(1) with quadratic trend model.

$$ARL = H(u) = \mathbf{E}_\infty(\tau_b) < \infty.$$

where $\tau_b = \inf\{t > 0; E_t > b\}, b > u.$

The function $H(u)$ is extended into the Fredholm Integral Equations of the second kind.

$$H(u) = 1 + \mathbf{E}_E\left[I\{0 < E_1 < b\}H(E_1)\right] + \mathbf{P}_E\{E_1 = 0\}H(0).$$

Where $\mathbf{P}_E$ represents the probability measure and $\mathbf{E}_E$ represents the expectation corresponding to the initial value of $E_0 = u.$

The EWMA statistics E_1 represents a state of being in-control, which can be visualized as equation (3)

$$0 \le (1-\lambda)E_0 + \lambda c + \lambda\phi Y_{t-1} + \lambda\beta_1 t + \lambda\beta_2 t^2 \le b. \quad (3)$$

In the event that Y_1 produces an out-of-control state for E_1, then

$$(1-\lambda)E_0 + \lambda c + \lambda\phi Y_{t-1} + \lambda\beta_1 t + \lambda\beta_2 t^2 > b. \quad (4)$$

or $(1-\lambda)E_0 + \lambda c + \lambda\phi Y_{t-1} + \lambda\beta_1 t + \lambda\beta_2 t^2 < 0.$

For an initial value $E_0 = u,$ then equation (4) can be rewritten as follows:

$$0 < (1-\lambda)u + \lambda c + \lambda\phi Y_{t-1} + \lambda\beta_1 t + \lambda\beta_2 t^2 < b.$$

Following Champ and Rigdon's method (1991), the initial value $E_0 = u,$ and $\varepsilon_t \sim Exp(\alpha).$ The function $H(u)$ can be rewritten as follows:

$$H(u)=1+\int_0^b H(E_1)f(\varepsilon_1)d\varepsilon_1. \quad (5)$$

$$=1+\int_0^b H((1-\lambda)u+\lambda c+\lambda\phi Y_{t-1}+\lambda\beta_1 t+\lambda\beta_2 t^2)f(z)dz.$$

Equation (5) is changed the variable of integration, then $H(u)$ is obtained as:

$$H(u)=1+\frac{1}{\lambda}\int_0^b H(z)f\left(\frac{z-(1-\lambda)u}{\lambda}-c-\phi Y_{t-1}-\beta_1 t-\beta_2 t^2\right)dz.$$

$$=1+\frac{1}{\lambda}\int_0^b H(z)\left(\frac{1}{\alpha}e^{-\frac{1}{\alpha}\left\{\frac{z-(1-\lambda)u}{\lambda}-c-\phi Y_{t-1}-\beta_1 t-\beta_2 t^2\right\}}\right)dz.$$

The Fredholm Integral Equation of the second kind can be derived as follows:

$$H(u)=1+\frac{1}{\lambda\alpha}\int_0^b H(z)e^{-\frac{z}{\lambda\alpha}}e^{\left\{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{(c+\phi Y_{t-1}+\beta_1 t+\beta_2 t^2)}{\alpha}\right\}}dz.$$

If we assume that $C(u)=e^{\left\{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{(c+\phi Y_{t-1}+\beta_1 t+\beta_2 t^2)}{\alpha}\right\}}$, then we can rewrite the equation as follows:

$$H(u)=1+\frac{C(u)}{\lambda\alpha}\int_0^b H(z)e^{-\frac{z}{\lambda\alpha}}dz,\ 0\le u\le b.$$

Let $k=\int_0^b H(z)e^{-\frac{z}{\lambda\alpha}}dz$, then we have

$$H(u)=1+\frac{C(u)}{\lambda\alpha}k. \quad (6)$$

For solving k we obtain

$$k=\int_0^b H(z)e^{-\frac{z}{\lambda\alpha}}dz$$

$$=\int_0^b\left(1+\frac{k}{\lambda\alpha}C(z)\right)e^{-\frac{z}{\lambda\alpha}}dz$$

$$=\int_0^b e^{-\frac{z}{\lambda\alpha}}dz+\int_0^b\frac{C(z)}{\lambda\alpha}ke^{-\frac{z}{\lambda\alpha}}dz$$

$$=\int_0^b e^{-\frac{z}{\lambda\alpha}}dz+\frac{k}{\lambda\alpha}\int_0^b C(z)e^{-\frac{z}{\lambda\alpha}}dz$$

$$=\int_0^b e^{-\frac{z}{\lambda\alpha}}dz+\frac{k}{\lambda\alpha}.e^{\frac{c+\phi Y_{t-1}+\beta_1 t+\beta_2 t^2}{\alpha}}\int_0^b e^{\frac{(1-\lambda)z}{\lambda\alpha}-\frac{z}{\lambda\alpha}}dz$$

$$=-\lambda\alpha(e^{-\frac{b}{\lambda\alpha}}-1)+\frac{k}{\lambda\alpha}.e^{\frac{c+\phi Y_{t-1}+\beta_1 t+\beta_2 t^2}{\alpha}}\int_0^b e^{-\frac{z}{\alpha}}dz$$

$$=-\lambda\alpha(e^{-\frac{b}{\lambda\alpha}}-1)-\frac{k}{\lambda}.e^{\frac{c+\phi Y_{t-1}+\beta_1 t+\beta_2 t^2}{\alpha}}(e^{\frac{-b}{\alpha}}-1).$$

As a result, the following formula can be utilized to determine a constant k:

$$k=\frac{-\lambda\alpha(e^{-\frac{b}{\lambda\alpha}}-1)}{1+\frac{1}{\lambda}.e^{\frac{c+\phi Y_{t-1}+\beta_1 t+\beta_2 t^2}{\alpha}}.(e^{-\frac{b}{\alpha}}-1)}. \quad (7)$$

Next, substitute equation (7) into equation (6) to derive the explicit formula of ARL for the AR(1) with quadratic trend model.

$$H(u)=1-\frac{\lambda e^{\frac{(1-\lambda)u}{\lambda\alpha}}(e^{-\frac{b}{\lambda\alpha}}-1)}{\lambda e^{-\frac{c+\phi Y_{t-1}+\beta_1 t+\beta_2 t^2}{\alpha}}+(e^{-\frac{b}{\alpha}}-1)}.$$

If the exponential parameter (α) is determined with α_0 before the start of the process, then the ARL is called ARL_0. Similarly, if the exponential parameter (α) is assigned to $\alpha_1=\alpha_0(1+\delta)$, where $\alpha_1>\alpha_0$ and δ is the shift sizes in an out-of-control process, then the ARL is called ARL_1.

The ARL solution demonstrates the uniqueness of the integral equation for explicit formulas using Banach's Fixed-point Theorem. The theorem applies to a metric space consisting of continuous functions on a closed interval $(C(I),\|\ \|_\infty)$ where I denote the compact interval. The norm $\|H\|_\infty=Sup_{u\in I}|H(u)|$ and the operator T are defined on this space. If there exists a number $0\le q<1$ such that the operator T is a contraction, then the theorem holds true

$$\|T(H_1)-T(H_2)\|_\infty\le q\|H_1-H_2\| \text{ for all } H_1,H_2\in I.$$

Let $C(I_1)$ as a continuous function on $I_1=[0,b]$ and define the operator T as

$$T(H(u))=1+\frac{1}{\lambda\alpha}e^{\left\{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{(c+\phi Y_{t-1}+\beta_1 t+\beta_2 t^2)}{\alpha}\right\}}\int_0^b H(z)e^{-\frac{z}{\lambda\alpha}}dz.$$

According to the Banach Fixed Point Theorem, if the operator T is a contraction, then fixed point equations $T(H(u))=H(u)$ has a unique solution. So, in this case, if T is a contraction, then the integral equation can be written as $T(H(u))=H(u)$, and it will have a unique solution.

Theorem: Banach's Fixed-point Theorem

In the complete metric space (X,d) where $T: X \to X$ is a mapping satisfying the criteria of a contraction mapping with contraction constant $q<1$ such that $\|T(H_1)-T(H_2)\|_\infty \le q\|H_1-H_2\|$, there is a unique function $H(\cdot)\in X$ for which $T(H(u))=H(u)$ has a unique fixed point in X.

Proof:

For any given $u \in I$ and $H_1, H_2 \in C(I)$, we have the inequality $\|T(H_1)-T(H_2)\|_\infty \le q\|H_1-H_2\|$ where q < 1. According to (11), we obtain

$$\|T(H_1)-T(H_2)\|_\infty = Sup_{u\in[0,b]}\left|H_1(z)-H_2(z)\right|$$

$$= Sup_{u\in[0,b]}\left|\frac{e^{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{c+\phi Y_{t-1}+\beta_1 t+\beta_2 t^2}{\alpha}}}{\lambda\alpha}\int_0^b \left(H_1(z)-H_2(z)\right)e^{-\frac{z}{\lambda\alpha}}dz\right|$$

$$\le \sup_{u\in[0,b]}\left\{\frac{e^{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{c+\phi Y_{t-1}+\beta_1 t+\beta_2 t^2}{\alpha}}}{\lambda\alpha}\left|\int_0^b e^{-\frac{z}{\lambda\alpha}}dz\right|\right\}\|H_1-H_2\|_\infty$$

$$\le \sup_{u\in[0,b]}\left\{e^{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{c+\phi Y_{t-1}+\beta_1 t+\beta_2 t^2}{\alpha}}\left(1-e^{-\frac{b}{\lambda\alpha}}\right)\right\}\|H_1-H_2\|_\infty$$

$$\le q\|H_1-H_2\|_\infty,$$

where $q=\sup_{u\in[0,b]}\left[e^{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{c+\phi Y_{t-1}+\beta_1 t+\beta_2 t^2}{\alpha}}\left(1-e^{-\frac{b}{\lambda\alpha}}\right)\right]<1.$

4. NUMERICAL INTEGRATION METHOD

The section addresses the numerical integration of ARL of the EWMA control chart for AR(1) with quadratic trend model. We will use Gauss-Legendre Quadrature rule to approximate the ARL integral equation. Consequently, the following $\tilde{H}(u)$ can be used to define the integral equation in (5).

$$\tilde{H}(u)=1+\frac{1}{\lambda}\int_0^b H(z)f\left(\frac{z-(1-\lambda)u}{\lambda}-c-\phi Y_{t-1}-\beta_1 t-\beta_2 t^2\right)dz.$$

The quadrature rule is a mathematical method that allows for the numerical integration of integral equations using finite sums. The approximation for an integral has the following form:

$$\int_0^b W(z)f(z)dz \approx \sum_{j=1}^m w_j f(a_j).$$

where $w_j=\frac{b}{m}$ and $a_j=\frac{b}{m}\left(j-\frac{1}{2}\right); j=1,2,\ldots,m.$

The numerical approximation $\tilde{H}(a_i)$ is used to solve the integral equation through a linear algebraic method.

$$\tilde{H}(a_i)\approx 1+\frac{1}{\lambda}\sum_{j=1}^m w_j\tilde{H}(a_j)f\left(\frac{a_j-(1-\lambda)a_i}{\lambda}-c-\phi Y_{t-1}-\beta_1 t-\beta_2 t^2\right).$$

Thus,

$$\tilde{H}(a_1)\approx 1+\frac{1}{\lambda}\sum_{j=1}^m w_j\tilde{H}(a_j)f\left(\frac{a_j-(1-\lambda)a_1}{\lambda}-c-\phi Y_{t-1}-\beta_1 t-\beta_2 t^2\right).$$

$$\tilde{H}(a_2)\approx 1+\frac{1}{\lambda}\sum_{j=1}^m w_j\tilde{H}(a_j)f\left(\frac{a_j-(1-\lambda)a_2}{\lambda}-c-\phi Y_{t-1}-\beta_1 t-\beta_2 t^2\right).$$

$$\vdots$$

$$\tilde{H}(a_m)\approx 1+\frac{1}{\lambda}\sum_{j=1}^m w_j\tilde{H}(a_j)f\left(\frac{a_j-(1-\lambda)a_m}{\lambda}-c-\phi Y_{t-1}-\beta_1 t-\beta_2 t^2\right).$$

It can be rewritten in matrix form.

$$\mathbf{H}_{m\times1}=(\mathbf{I}_m-\mathbf{R}_{m\times m})^{-1}\mathbf{1}_{m\times1}.$$

where $\mathbf{H}_{m\times1}=\begin{pmatrix}\tilde{H}(a_1)\\ \tilde{H}(a_2)\\ \vdots\\ \tilde{H}(a_m)\end{pmatrix}$, $\mathbf{1}_{m\times1}=\begin{pmatrix}1\\1\\ \vdots\\1\end{pmatrix}$.

and $[\mathbf{R}]_{ij}\approx\frac{1}{\lambda}w_j f\left(\frac{a_j-(1-\lambda)a_i}{\lambda}-c-\phi Y_{t-1}-\beta_1 t-\beta_2 t^2\right).$

The numerical integral equation for the ARL of the EWMA control chart is provided below.

$$\tilde{H}(u)\approx 1+\frac{1}{\lambda}\sum_{j=1}^m w_j\tilde{H}(a_j)f\left(\frac{a_j-(1-\lambda)u}{\lambda}-c-\phi Y_{t-1}-\beta_1 t-\beta_2 t^2\right).$$

5. NUMERICAL RESULTS

The study analyzed the explicit formulas and the NIE approach used for detecting changes in the process mean on an EWMA control chart for AR(1) with a quadratic trend model. It compared the findings obtained through numerical integration, especially the absolute percentage difference, with the accuracy and reliability of ARL values calculated from explicit formulas. The absolute percentage difference of ARL can be calculated by

$$Diff\,(\%)=\frac{\left|H(u)-\tilde{H}(u)\right|}{H(u)}\times100.$$

Tables 1 through 6 display the numerical results for ARL_0 = 370. The ARL_1 of the EWMA control chart for the AR(1) with a quadratic trend model was analyzed using two different methods: the explicit formula and the NIE method. The analysis used an EWMA control chart and determined an exponential smoothing parameter ($\lambda=0.10, 0.15$). In-control state: exponential parameter of white noise process (α_0) = 1. Out-of-control state: (α_1) ranges from 1.01, 1.03, 1.05, 1.10, 1.20, 1.30, 1.40, 1.50 and 2.00, respectively. The study examined AR(1) with quadratic trend model when the autoregressive coefficient parameter (ϕ) = 0.10, 0.20 and 0.30. The linear (β_1) and the quadratic (β_2) coefficients parameter; ($\beta_1 = 0.20, \beta_2 = 0.30$), ($\beta_1 = 0.30, \beta_2 = 0.50$), and ($\beta_1 = 0.50$, $\beta_2 = 0.80$), respectively. The comparison analysis showed that the ARL values obtained by the explicit formula and the NIE methods were highly similar. It is important to mention that the absolute percentage difference of ARL calculated resulted in a value of zero, which suggests that there was no noticeable difference between the ARL values derived from both methods. However, the CPU time of the explicit formula is faster than the NIE method by around 1-2 seconds.

Table 1 ARL for the explicit formulas and the NIE method on EWMA control chart for AR(1) with quadratic trend model with $\alpha_0 = 1$, $\lambda = 0.10, \phi = 0.10$, $\beta_1 = 0.20, \beta_2 = 0.30$, $b = 0.00242$ and ARL_0 = 370.283

α_1	Explicit		NIE	
	ARL	CPU time	ARL	CPU time
1.01	333.273	0.001	333.273	2.199
1.03	271.597	0.001	271.597	2.247
1.05	223.023	0.001	223.023	2.278
1.10	140.524	0.001	140.524	2.184
1.20	62.5586	0.001	62.5586	2.246
1.30	31.6155	0.001	31.6155	2.152
1.40	17.7351	0.001	17.7351	2.278
1.50	10.8692	0.001	10.8692	2.309
2.00	2.48567	0.001	2.48567	2.246

Table 2 ARL for the explicit formulas and the NIE method on EWMA control chart for AR(1) with quadratic trend model with $\alpha_0 = 1, \lambda = 0.15, \phi = 0.10$, $\beta_1 = 0.20, \beta_2 = 0.30$, $b = 0.05016$ and ARL_0 = 370.006

α_1	Explicit		NIE	
	ARL	CPU time	ARL	CPU time
1.01	337.264	0.001	337.264	2.309
1.03	282.224	0.001	282.224	2.231
1.05	238.224	0.001	238.224	2.230
1.10	161.300	0.001	161.300	2.324
1.20	82.9552	0.001	82.9552	2.231
1.30	47.9105	0.001	47.9105	2.247
1.40	30.1893	0.001	30.1893	2.138
1.50	20.3680	0.001	20.3680	2.325
2.00	5.54718	0.001	5.54718	2.262

Table 3 ARL for the explicit formulas and the NIE method on EWMA control chart for AR(1) with quadratic trend model with $\alpha_0 = 1, \lambda = 0.10, \phi = 0.20$, $\beta_1 = 0.30, \beta_2 = 0.50$, $b =$ 0.001615 and ARL_0 = 370.059

α_1	Explicit		NIE	
	ARL	CPU time	ARL	CPU time
1.01	331.688	0.001	331.688	0.609
1.03	268.137	0.001	268.137	0.578
1.05	218.49	0.001	218.49	0.531
1.10	135.226	0.001	135.226	0.438
1.20	58.3912	0.001	58.3912	0.406
1.30	28.8007	0.001	28.8007	0.328
1.40	15.8592	0.001	15.8592	0.453
1.50	9.59491	0.001	9.59491	0.344
2.00	2.20971	0.001	2.20971	0.500

Table 4 ARL for the explicit formulas and the NIE method on EWMA control chart for AR(1) with quadratic trend model with $\alpha_0 = 1, \lambda = 0.15, \phi = 0.20$, $\beta_1 = 0.30, \beta_2 = 0.50$, $b =$ 0.03270 and ARL_0 = 370.097

α_1	Explicit		NIE	
	ARL	CPU time	ARL	CPU time
1.01	334.368	0.001	334.368	1.890
1.03	275.320	0.001	275.320	1.843
1.05	229.104	0.001	229.104	1.781
1.10	150.532	0.001	150.532	1.828
1.20	74.1248	0.001	74.1248	1.953
1.30	41.5329	0.001	41.5329	1.890
1.40	25.6008	0.001	25.6008	1.828
1.50	16.9977	0.001	16.9977	2.328
2.00	4.52274	0.001	4.52274	1.812

Table 5 ARL for the explicit formulas and the NIE method on EWMA control chart for AR(1) with quadratic trend model with $\alpha_0 = 1, \lambda = 0.10, \phi = 0.30$, $\beta_1 = 0.50, \beta_2 = 0.80$, $b =$ 0.000884 and ARL_0 = 370.395

α_1	Explicit		NIE	
	ARL	CPU time	ARL	CPU time
1.01	329.93	0.001	329.93	1.859
1.03	263.526	0.001	263.526	1.765
1.05	212.276	0.001	212.276	1.625
1.10	127.923	0.001	127.923	1.734
1.20	52.7900	0.001	52.7900	1.875
1.30	25.1230	0.001	25.1230	1.859
1.40	13.4700	0.001	13.4700	1.891
1.50	8.00802	0.001	8.00802	1.875
2.00	1.89231	0.001	1.89231	1.750

Table 6 ARL for the explicit formulas and the NIE method on EWMA control chart for AR(1) with quadratic trend model with $\alpha_0 = 1, \lambda = 0.15, \phi = 0.30, \beta_1 = 0.50, \beta_2 = 0.80, b =$ 0.01750 and ARL_0 = 370.087

α_1	Explicit		NIE	
	ARL	CPU time	ARL	CPU time
1.01	330.059	0.001	330.059	2.000
1.03	265.537	0.001	265.537	1.843
1.05	216.521	0.001	216.521	1.828
1.10	136.464	0.001	136.464	1.687
1.20	63.3354	0.001	63.3354	1.875
1.30	34.0676	0.001	34.0676	1.766
1.40	20.3922	0.001	20.3922	2.172
1.50	13.2606	0.001	13.2606	1.906
2.00	3.46240	0.001	3.46240	1.859

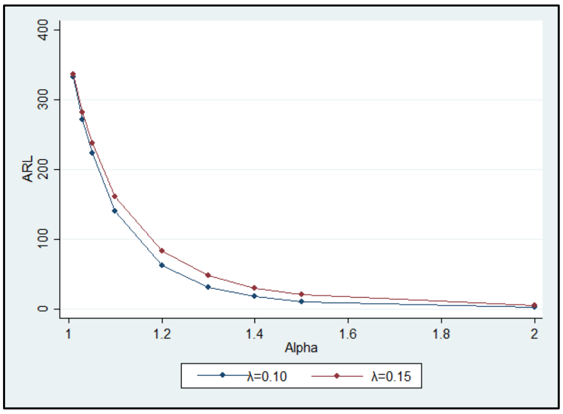

Figure 1 Comparison of ARL_1 of EWMA control charts under AR(1) with quadratic trend model ($\phi = 0.10, \beta_1 = 0.20, \beta_2 = 0.30$) when $\lambda = 0.10$ and 0.15

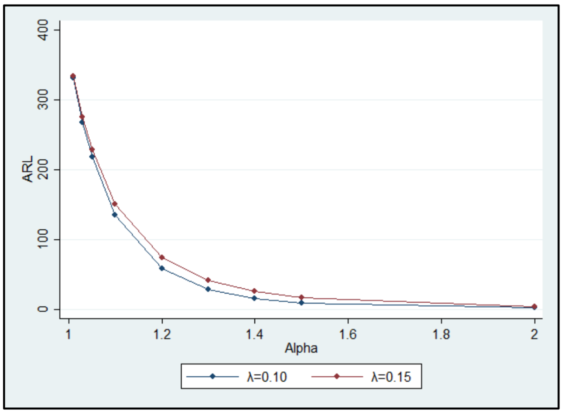

Figure 2 Comparison of ARL_1 of EWMA control charts under AR(1) with quadratic trend model ($\phi = 0.20, \beta_1 = 0.30, \beta_2 = 0.50$) when $\lambda = 0.10$ and 0.15

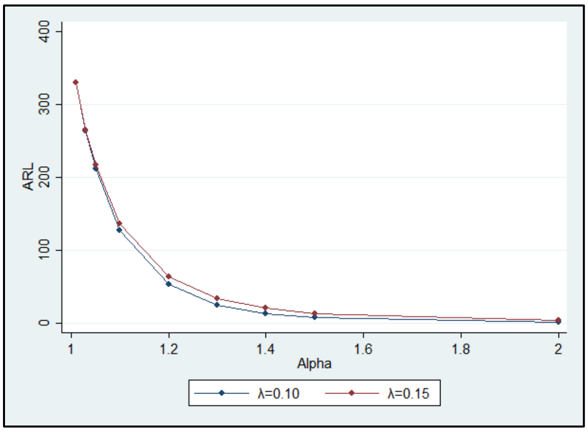

Figure 3 Comparison of ARL_1 of EWMA control charts under AR(1) with quadratic trend model ($\phi = 0.30, \beta_1 = 0.50, \beta_2 = 0.80$) when $\lambda = 0.10$ and 0.15

6. DISCUSSION

The study aimed to calculate the ARL value to detect process mean shifts in an AR(1) with quadratic trend model, using explicit formulas and the NIE method. Establishing the existence and uniqueness of the ARL is crucial for ensuring the accuracy and reliability of any statistical analysis. The validation of explicit formulas has proven beyond doubt that the ARL can be calculated with precision, providing a solid foundation for making informed decisions and driving positive outcomes. This study provides valuable insights into the effectiveness of ARL analysis for monitoring mean shifts. By comparing explicit formulas and the NIE method, the study highlights the benefits and limitations of each approach. This research can help organizations make informed decisions about which method to use in their own monitoring processes. A numerical investigation was conducted to determine the in-control ARL for different parameter configurations and levels of process mean shift. The results showed that there was no significant difference between the absolute percentage difference of the ARLs derived from explicit formulas and those obtained through the NIE method. These findings suggest that either method can be used with confidence to calculate ARLs. Although both methods produced similar results, we observed that the explicit formulas exhibited a significant advantage in computational efficiency. The explicit formulas required considerably less processing time compared to the NIE method. Hence, if you are working on a project that requires speedy calculations, then the explicit formulas may be the more practical option to use.

ACKNOWLEDGMENT(S)

The authors gratefully acknowledge the editor and referees for their valuable comments and suggestions which greatly improve this paper. The research was funding by Applied Science Faculty King Mongkut's University of Technology North Bangkok Contract no. 672187

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