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Wave solutions to the combined KdV-mKdV equation via two methods with the Riccati equation

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ABSTRACT

The exact traveling wave solutions of the combined KdV-mKdV equation, which are the partial differential equations, were examined using the simple equation method with the Riccati equation and the modified extended tanh-function method. The solutions of the combined KdV-mKdV equation are obtained in terms of hyperbolic functions and trigonometric functions. Some solutions are created in the form of kink waves, which are represented by the two-dimensional graph, the three-dimensional graph, and the contour graph. Moreover, the results validated that the methods used in this study were powerful mathematical tools to find exact wave solutions to nonlinear models encountered in various areas of science and engineering.

KEYWORDS: the simple equation method with the Riccati equation, the modified extended tanh-function method, the combined KdV-mKdV equation, partial differential equation.

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1. INTRODUCTION

In many branches of natural science, accurate solutions of non-linear partial differential equations (NLPDEs) are essential to comprehending the qualitative aspects of numerous processes and phenomena. In the present, researchers have discovered numerous potent techniques for resolving the nonlinear partial differential equation, such as the simple equation method (Sanjun & Chankaew, 2022), the tanh-coth method (Kumar & Pankaj, 2015), the sin-cosine method (Raslan et al., 2017), the (G′/G)-expansion method (Akbar et al., 2018), the modified extended tanh-function method (Zahran & Khater, 2016), the Exp-expansion method (He & Wu, 2006), the unified method (Abdel-Gawad et al., 2022), the Riccati-Bernoulli sub-ODE method (Yang et al., 2015), etc.

The combined KdV-mKdV equation (Khan et al., 2023),

$$u_t + muu_x + nu^2u_x + u_{xxx} = 0, \qquad (1)$$

where $u = u(x,t)$, m and n are real constants, which is a useful tool in the study of water waves. It helps researchers understand and forecast the actions of many wave phenomena, such as solitary waves, wave breaking, turbulence, interactions between waves and structures, and tsunamis (Yuan et al., 2023). Many analytical methods have been used to investigate this equation. from multiple authors, such as in 2010 (Lu & Shi, 2010) using the expansion approach of Jacobi elliptic functions; in 2012 (Naher & Abdullah, 2012) using the improved (G′/G)-expansion method; in 2014 (Huang et al., 2014) using the complex method; in 2016 (Hu et al., 2016) using the consistent tanh expansion (CTE) method; in 2022 (Ekici & Ünal, 2022) using the rational (G′/G)-expansion method (Ekici & Ünal, 2022); and in 2023 using the Bernoulli sub-ODE method (Khan et al., 2023).

Two different methods are used in this article. The first method is called the simple equation method with Riccati equation, which was first proposed to construct exact traveling wave solutions of the Kodomtsev-Petviashvili (KP) equation, the (2 + 1)-dimensional breaking soliton equation, and the modified generalized Vakhnenko equation in 2016 (Nofal, 2016). In 2024, (Thadee & Phoosree, 2024) used the simple equation method with the Riccati equation to find the exact traveling wave solutions of fourth-order fractional water wave equations. And the modified extended tanh-function method that executes this method has broad applicability to many other nonlinear evolution equations. In mathematical physics, the modified extended tanh-function method was used to investigate exact solutions to various equations, such as the Hirota-Satsuma-coupled KdV system in 2007 (El-Wakil & Abdou, 2007). The fisher-type equation, the ZK-BBM equation, generalized the Burgers-Fisher equation in 2007 (El-Wakil & Abdou, 2007). The Bogoyavlenskii equation in 2016 (Zahran & Khater, 2016). And the conformable time fractional Drinfel'd-Sokolov-Wilson equation (Bashar et al., 2023).

In this work, we solve the combined KdV-mKdV equation using the simple equation method with the Riccati equation and the modified extended tanh-function method for finding the traveling wave solutions of the combined KdV-mKdV equation. We have shown the new exact solutions and obtained the wave solutions for the wave effects in a two-dimensional graph, a three-dimensional graph, and a contour graph in Section. 4. Moreover, we compared the solutions and got the new simpler form,which is shown in Section. 5. Finally, Section. 6 concludes with observations and results.

2. MATERIALS AND METHODS

Let there be a NLPDE, say, in two independent variables x and t, is given by:

$$G\left(u,u_t,u_x,u_{xx},u_{xt},\ldots\right)=0, \tag{2}$$

where G is in general a polynomial function of $u\left(x,t\right)$ and its arguments; the subscripts denote the partial derivatives. Start by considering combining the independent variables x and t into one variable, ξ. We suppose that

$$u\left(x,t\right)=u\left(\xi\right),\ \xi=x-\omega t. \tag{3}$$

The traveling wave transformation Eq. (3) permits us to reduce Eq. (2) to the following ordinary differential equation (ODE):

$$Q\left(u,u',u'',u''',\ldots\right)=0, \tag{4}$$

where Q is a polynomial in $u\left(\xi\right)$ and its total derivatives, where $u'\left(\xi\right)=\frac{du}{d\xi}$, $u''\left(\xi\right)=\frac{d^2u}{d\xi^2}$, and so on.

2.1 The simple equation method with the Riccati equation

We outline the fundamental steps of the simple equation method with the Riccati equation (Nofal, 2016) as follows:

Step 1. Start by considering Eqs. (2)-(4).

Step 2. Suppose that the solution of Eq. (4) is in the following form:

$$u\left(\xi\right)=\sum_{i=0}^{N}a_iZ^i\left(\xi\right). \tag{5}$$

Which $a_i\left(i=0,1,2,\ldots,N\right)$ are constants that need to be determined such that $a_N\neq 0$ and $Z\left(\xi\right)$ conform to the following the Riccati equation,

$$Z'\left(\xi\right)=\alpha Z^2\left(\xi\right)+\beta, \tag{6}$$

where α and β are non-zero constants. The two-case solutions of Eq. (6) can be explained here:

Case 1: $\alpha\beta < 0$,

$$z(\xi) = -\frac{\sqrt{-\alpha\beta}}{\alpha}\tanh\left(\sqrt{-\alpha\beta}\,\xi - \frac{V\ln(\xi_0)}{2}\right), \tag{7}$$

where $\xi_0 > 0$ and $V = \pm 1$.

Case 2: $\alpha\beta > 0$,

$$z(\xi) = \frac{\sqrt{\alpha\beta}}{\alpha}\tan\left(\sqrt{\alpha\beta}\,(\xi + \xi_0)\right), \tag{8}$$

where ξ_0 is a constant.

Step 3. The balance number N may be achieved by striking a balance between the derivative of the highest-order and the highest nonlinear terms that exist in Eq. (4).

Step 4. For the terms that were all in the same power of z, we added up all of the coefficients and set them to zero. We obtained $\omega, \alpha\beta$ and a_i. Thus, the solutions to Eq. (2) that include traveling waves are constructed.

2.2 The modified extended tanh-function method

We describe the main steps of the modified extended tanh-function method (Zahran & Khater, 2016).

Step 1. Start by considering Eqs. (2)-(4).

Step 2. Suppose that the solution of Eq. (4) in the following form:

$$u(\xi) = a_0 + \sum_{i=1}^{N}\left(a_i\varphi^{i} + b_i\varphi^{-i}\right), \tag{9}$$

where a_i and b_i are constants that need to be determined such that $a_N \neq 0$ or $b_N \neq 0$ and φ satisfy the Riccati equation,

$$\varphi' = \sigma + \varphi^2 \tag{10}$$

where σ is a constant. Eq. (6) admits several types of solutions according to the following:

Case 1: If $\sigma < 0$, then

$$\varphi = -\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\,\xi\right), \tag{11}$$

or

$$\varphi = -\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\,\xi\right). \tag{12}$$

Case 2: If $\sigma > 0$, then

$$\varphi = \sqrt{\sigma}\tan\left(\sqrt{\sigma}\,\xi\right), \tag{13}$$

or

$$\varphi = -\sqrt{\sigma}\cot\left(\sqrt{\sigma}\,\xi\right). \tag{14}$$

Case 3: If $\sigma = 0$, then

$$\varphi = -\frac{1}{\xi}. \tag{15}$$

Step 3. The balance number N may be achieved by striking a balance between the derivative of the highest-order and the highest nonlinear terms that exist in Eq. (4).

Step 4. Substitute Eq. (9) and its derivative as well as Eq. (10) into Eq. (4). Then by setting the coefficients of $\varphi^i, (i = 0, \pm 1, \pm 2, \ldots)$, and equating them to zero, we obtain a system of algebraic equations, which can be solved to obtain the values of a_i, b_i, σ and ω.

Step 5. Substitute the values of a_i, b_i, σ, ω and the solutions of Eq. (10) into Eq. (9) and we obtain the exact solutions of Eq. (2).

3. APPLICATIONS

We next want to solve the combined KdV-mKdV equation using the simple equation method with the Riccati equation and the modified extended tanh-function method.

The combined KdV-mKdV equation is

$$u_t + muu_x + nu^2u_x + u_{xxx} = 0, \tag{16}$$

where $u = u(x,t)$, m and n are real constants. Using $u(\xi) = u(x,t)$ and the traveling wave variable

$\xi = x - \omega t$, we will transform it into an ODE. When the transformation is put into equation (16), the outcome is

$$-\omega u' + muu' + nu^2 u' + u''' = 0. \quad (17)$$

Integrating Eq. (17) with the zero constant, we get:

$$-\omega u + \frac{m}{2}u^2 + \frac{nu^3}{3} + u'' = 0. \quad (18)$$

The proposed procedures are used in the following sections to achieve the desired results.

3.1 Solutions with the simple equation method with the Riccati equation

Next, we balanced the highest-order derivative terms u'' with the highest nonlinear terms u^3 in Eq. (18). Then $N = 1$. We have the solution to Eq. (18) as follows:

$$u(\xi) = a_0 + a_1 Z, \quad (19)$$

where Z satisfies Eq. (6). Therefore, the expressions for u'', u^2 and u^3 are expressed as:

$$u'' = 2\alpha^2 a_1 Z^3 + 2\alpha\beta a_1 Z, \quad (20)$$

$$u^2 = a_0^2 + 2a_0 a_1 Z + a_1^2 Z^2, \quad (21)$$

$$u^3 = a_0^3 + 3a_0^2 a_1 Z + 3a_0 a_1^2 Z^2 + a_1^3 Z^3. \quad (22)$$

Substituting Eqs. (19)-(22) into Eq. (18), the outcome is

$$\left(-\omega a_0 + \frac{m}{2}a_0^2 + \frac{n}{2}a_0^3\right) + \left(-\omega a_1 + ma_0 a_1 + na_0^2 a_1 + 2\alpha\beta a_1\right)Z + \left(\frac{m}{2}a_1^2 + na_0 a_1^2\right)Z^2 + \left(\frac{n}{3}a_1^3 + 2\alpha^2 a_1\right)Z^3 = 0. \quad (23)$$

Then we set each coefficient of Z^i to zero, where $i = 0, 1, 2, 3,$ yields

$$Z^0(\xi):\ -\omega a_0 + \frac{m}{2}a_0^2 + \frac{n}{2}a_0^3 = 0, \quad (24)$$

$$Z^1(\xi):\ -\omega a_1 + ma_0 a_1 + na_0^2 a_1 + 2\alpha\beta a_1 = 0, \quad (25)$$

$$Z^2(\xi):\ \frac{m}{2}a_1^2 + na_0 a_1^2 = 0, \quad (26)$$

$$Z^3(\xi):\ \frac{n}{3}a_1^3 + 2\alpha^2 a_1 = 0. \quad (27)$$

After solving this collection of mathematical equations, we get

$$a_0 = \frac{-m}{2n},\ a_1 = \pm\alpha\sqrt{\frac{-6}{n}},\ \omega = \frac{-m^2}{8n} \text{ and } \alpha\beta = \frac{m^2}{16n}. \quad (28)$$

By Eqs. (7), (8), (28) and $\xi = x - \omega t$, the exact solutions of the combined KdV-mKdV equation are shown for two cases with an arbitrary constant ξ_0.

Case 1: $\alpha\beta < 0$,

$$u_1(x,t) = -\frac{m}{2n} + \frac{m\sqrt{6}}{4n}\tanh\left(\sqrt{-\frac{m^2}{16n}}\left(x + \left(\frac{m^2}{8n}\right)t\right) - \frac{V\ln\xi_0}{2}\right), \quad (29)$$

$$u_2(x,t) = -\frac{m}{2n} - \frac{m\sqrt{6}}{4n}\tanh\left(\sqrt{-\frac{m^2}{16n}}\left(x + \left(\frac{m^2}{8n}\right)t\right) - \frac{V\ln\xi_0}{2}\right), \quad (30)$$

where $\xi_0 > 0$ and $V = \pm 1$.

Case 2: $\alpha\beta > 0$,

$$u_3(x,t) = \frac{-m}{2n} + \frac{m\sqrt{-6}}{4n}\tan\sqrt{\frac{m^2}{16n}}\left(x + \left(\frac{m^2}{8n}\right)t\right), \quad (31)$$

$$u_4(x,t) = \frac{-m}{2n} - \frac{m\sqrt{-6}}{4n}\tan\sqrt{\frac{m^2}{16n}}\left(x + \left(\frac{m^2}{8n}\right)t\right), \quad (32)$$

where ξ_0 is a constant.

3.2 Solutions with the modified extended tanh function method

According to the procedure that will be described, the balancing number N is a positive integer, which can be defined by balancing the highest-order derivative terms u'' with the highest nonlinear terms u^3 in Eq. (18). Thus, $N=1$. We have the solution to Eq. (18) as follows:

$$u(\xi)=a_0+a_1\varphi+b_1\varphi^{-1} \quad (33)$$

where φ satisfies Eq. (10). Therefore, the expressions for u'',u^2 and u^3 are expressed as:

$$u''(\xi)=2\sigma a_1\varphi+2a_1\varphi^3+2\sigma^2 b_1\varphi^{-3}+2\sigma b_1\varphi^{-1}, \quad (34)$$

$$u^2(\xi)=a_0^{\ 2}+2a_0a_1\varphi+2a_0b_1\varphi^{-1}+a_1^{\ 2}\varphi^2+2a_1b_1+b_1^{\ 2}\varphi^{-2}, \quad (35)$$

$$u^3(\xi)=a_0^{\ 3}+3a_0^2a_1\varphi+3a_0^2b_1\varphi^{-1}+3a_0a_1^2\varphi^2+6a_0a_1b_1+3a_0b_1^2\varphi^{-2}+3a_1^2b_1\varphi+3a_1b_1^2\varphi^{-1}+a_1^3\varphi^3+b_1^3\varphi^{-3}. \quad (36)$$

Substituting Eqs. (33)-(36) into Eq. (18), the outcome is

$$\left(-\omega a_0+\frac{m}{2}a_0^2+ma_1b_1+\frac{n}{3}a_0^3+2na_0a_1b_1\right)+\left(-\omega a_1+ma_0a_1+na_0^1a_1+na_1^2b_1+2\sigma a_1\right)\varphi+\left(-\omega b_1+ma_0b_1+na_0^2b_1+na_1b_1^2+2\sigma b_1\right)\varphi^{-1}+\left(\frac{m}{2}a_1^2+na_0a_1^2\right)\varphi^2+\left(\frac{m}{2}b_1^2+na_0b_1^2\right)\varphi^{-2}+\left(\frac{n}{3}a_1^3+2a_1\right)\varphi^3+\left(\frac{n}{3}b_1^3+2\sigma^2b_1\right)\varphi^{-3}=0. \quad (37)$$

Then we set each coefficient of φ^i to zero, where $i=0,\pm1,\pm2,\pm3$, yields

$$\varphi^0;-\omega a_0+\frac{m}{2}a_0^2+ma_1b_1+\frac{n}{3}a_0^3+2na_0a_1b_1=0, \quad (38)$$

$$\varphi^1;-\omega a_1+ma_0a_1+na_0^2a_1+na_1^2b_1+2\sigma a_1=0, \quad (39)$$

$$\varphi^{-1};-\omega b_1+ma_0b_1+na_0^2b_1+na_1b_1^2+2\sigma b_1=0, \quad (40)$$

$$\varphi^2;\frac{m}{2}a_1^2+na_0a_1^2=0, \quad (41)$$

$$\varphi^{-2};\frac{m}{2}b_1^2+na_0b_1^2=0, \quad (42)$$

$$\varphi^3;\frac{n}{3}a_1^3+2a_1=0, \quad (43)$$

$$\varphi^{-3};\frac{n}{3}b_1^3+2\sigma b_1=0. \quad (44)$$

After we solve this set of algebraic equations, we get

$$a_0=\frac{-m}{2n},a_1=\sqrt{\frac{-6}{n}},b_1=-\sigma\sqrt{\frac{-6}{n}},\sigma=\frac{m^2}{96n} \text{ and } \omega=-\frac{m^2}{6n}, \quad (45)$$

or

$$a_0=\frac{-m}{2n},a_1=-\sqrt{\frac{-6}{n}},b_1=\sigma\sqrt{\frac{-6}{n}},\sigma=\frac{m^2}{96n} \text{ and } \omega=-\frac{m^2}{6n}. \quad (46)$$

By Eqs. (11)-(14), (45)-(46), and $\xi=x-\omega t$, the exact solutions of the combined KdV-mKdV equation can be classified into the following cases according to Section 2.2:

Case 1: If $\sigma<0$, then

$$u_5(x,t)=\frac{-m}{2n}\left(1+\frac{1}{2}\tanh\sqrt{\frac{-m^2}{96n}}\left(x+\frac{m^2}{6n}t\right)+\frac{1}{2\tanh\sqrt{\frac{-m^2}{96n}}\left(x+\frac{m^2}{6n}t\right)}\right), \quad (47)$$

$$u_6(x,t)=\frac{-m}{2n}\left(1-\frac{1}{2}\tanh\sqrt{\frac{-m^2}{96n}}\left(x+\frac{m^2}{6n}t\right)-\frac{1}{2\tanh\sqrt{\frac{-m^2}{96n}}\left(x+\frac{m^2}{6n}t\right)}\right), \quad (48)$$

$$u_7(x,t)=\frac{-m}{2n}\left(1+\frac{1}{2}\coth\sqrt{\frac{-m^2}{96n}}\left(x+\frac{m^2}{6n}t\right)+\frac{1}{2\coth\sqrt{\frac{-m^2}{96n}}\left(x+\frac{m^2}{6n}t\right)}\right), \quad (49)$$

$$u_8(x,t)=\frac{-m}{2n}\left(1-\frac{1}{2}\coth\sqrt{\frac{-m^2}{96n}}\left(x+\frac{m^2}{6n}t\right)-\frac{1}{2\coth\sqrt{\frac{-m^2}{96n}}\left(x+\frac{m^2}{6n}t\right)}\right). \quad (50)$$

Case 2: If $\sigma > 0$, then

$$u_9(x,t) = \frac{-m}{2n}\left(1 + \frac{I}{2}\tan\sqrt{\frac{m^2}{96n}}\left(x + \frac{m^2}{6n}t\right) - \frac{I}{2\tan\sqrt{\frac{m^2}{96n}}\left(x + \frac{m^2}{6n}t\right)}\right), \quad (51)$$

$$u_{10}(x,t) = \frac{-m}{2n}\left(1 - \frac{I}{2}\tan\sqrt{\frac{m^2}{96n}}\left(x + \frac{m^2}{6n}t\right) + \frac{I}{2\tan\sqrt{\frac{m^2}{96n}}\left(x + \frac{m^2}{6n}t\right)}\right), \quad (52)$$

$$u_{11}(x,t) = \frac{-m}{2n}\left(1 + \frac{I}{2}\cot\sqrt{\frac{m^2}{96n}}\left(x + \frac{m^2}{6n}t\right) - \frac{I}{2\cot\sqrt{\frac{m^2}{96n}}\left(x + \frac{m^2}{6n}t\right)}\right), \quad (53)$$

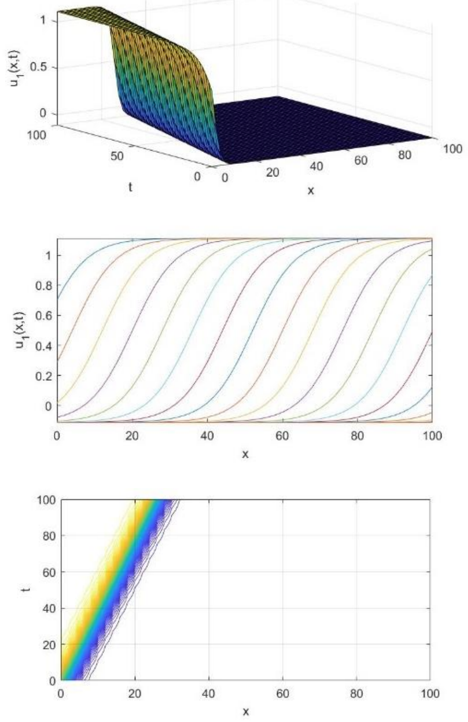
$$u_{12}(x,t) = \frac{-m}{2n}\left(1 - \frac{I}{2}\cot\sqrt{\frac{m^2}{96n}}\left(x + \frac{m^2}{6n}t\right) + \frac{I}{2\cot\sqrt{\frac{m^2}{96n}}\left(x + \frac{m^2}{6n}t\right)}\right). \quad (54)$$

4. GRAPHICAL REPRESENTATION OF SOME OBTAINED SOLUTIONS

The physical graphs of some solutions to the combined KdV-mKdV equation are shown in this section.

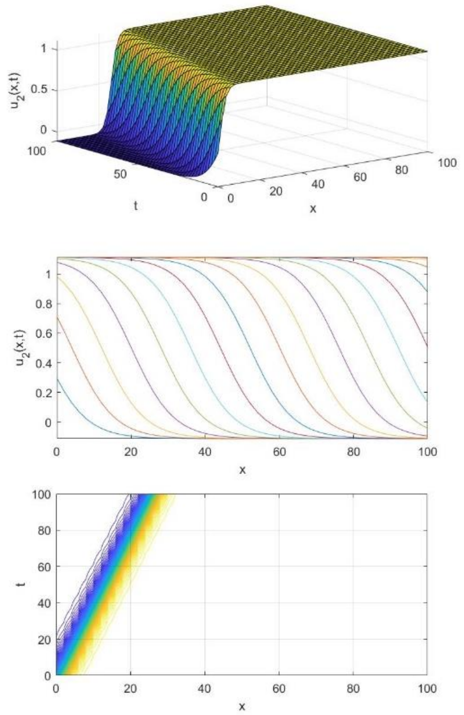
4.1 Graphical representation of the combined KdV-mKdV equation with the simple equation method with the Riccati equation

We set some parameters to get the example graph of the wave effects of the combined KdV-mKdV equation by $m = 2, n = -2, \xi_0 = 2$ in the interval $0 \le x, t \le 100$, which is displayed in Figures 1 and 2. It produces a kink wave solution.



$$u_1(x,t) = -\frac{m}{2n} + \frac{m\sqrt{6}}{4n}\tanh\left(\sqrt{-\frac{m^2}{16n}}\left(x + \left(\frac{m^2}{8n}\right)t\right) - \frac{\ln\xi_0}{2}\right).$$

Figures 1 Kink wave solution of $u_1(x,t)$ in 3D, 2D, and contour

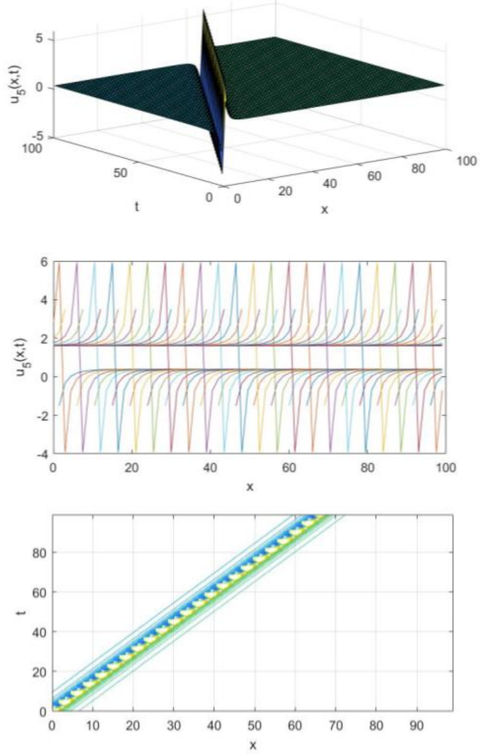


$$u_2(x,t) = -\frac{m}{2n} - \frac{m\sqrt{6}}{4n}\tanh\left(\sqrt{-\frac{m^2}{16n}}\left(x + \left(\frac{m^2}{8n}\right)t\right) - \frac{\ln\xi_0}{2}\right).$$

Figures 2 Kink wave solution of $u_2(x,t)$ in 3D, 2D, and contour.

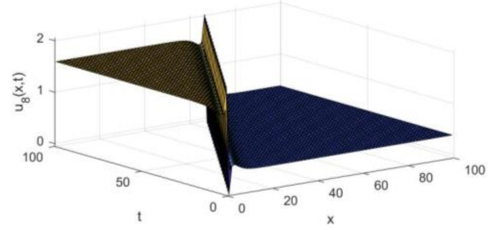
4.2 Graphical representation of the combined KdV-mKdV equation with the modified extended tanh function method

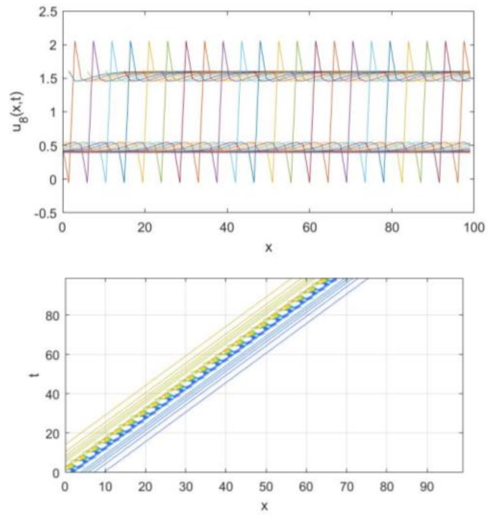
Next, we represent the shape of some solution to the combined KdV-mKdV equation with the modified extended tanh function method by setting some parameters $m=2$, $n=-1$ in the interval $0\leq x,t\leq 100$, which is displayed in Figures 3 and 4. It produces a kink wave solution.



$$u_5(x,t)=\frac{-m}{2n}\left(1+\frac{1}{2}\tanh\sqrt{\frac{-m^2}{96n}}\left(x+\frac{m^2}{6n}t\right)+\frac{1}{2\tanh\sqrt{\frac{-m^2}{96n}}\left(x+\frac{m^2}{6n}t\right)}\right),$$

Figures 3 Kink wave solution of $u_5(x,t)$ in 3D, 2D, and contour.





$$u_8(x,t)=\frac{-m}{2n}\left(1-\frac{1}{2}\coth\sqrt{\frac{-m^2}{96n}}\left(x+\frac{m^2}{6n}t\right)-\frac{1}{2\coth\sqrt{\frac{-m^2}{96n}}\left(x+\frac{m^2}{6n}t\right)}\right).$$

Figures 4. Kink wave solution of $u_8(x,t)$ in 3D, 2D, and contour.

5. SOLUTIONS COMPARISON

In this section, the solutions of the combined KdV-mKdV equation by the simple equation method with the Riccati equation and the modified extended tanh function method can be expressed in a simpler form than the rational (G′/G)-expansion method (Ekici & Ünal, 2022), as in Tables 1 and 2.

Table 1 Solutions comparison of the combined KdV-mKdV equation between the rational (G′/G)-expansion method and the simple equation method with the Riccati equation.

The rational (G′/G)-expansion method
Case 1: $\lambda^2-4\mu>0$
$u(\xi)=\pm\frac{1}{2}\sqrt{-\frac{6(\lambda^2-4\mu)}{\beta}}\left(\frac{c_1\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+c_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)}{c_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+c_2\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)}\right)$ $\pm\frac{\lambda}{2}\sqrt{-\frac{6}{\beta}}-\frac{\alpha}{2\beta}\mp\frac{3\lambda}{\beta}\sqrt{-\frac{\beta}{6}}$
Case 2: $\lambda^2-4\mu<0$
$u(\xi)=\pm\frac{1}{2}\sqrt{-\frac{6(4\mu-\lambda^2)}{\beta}}\left(\frac{-c_1\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)+c_2\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}{c_1\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)+c_2\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}\right)$ $\pm\frac{\lambda}{2}\sqrt{-\frac{6}{\beta}}-\frac{\alpha}{2\beta}\mp\frac{3\lambda}{\beta}\sqrt{-\frac{\beta}{6}}$

Case 3: $\lambda^2-4\mu=0$

$$u(\xi)=\pm\sqrt{-\frac{6}{\beta}}\left(\frac{c_2}{c_1+c_2\xi}\right)\mp\frac{\lambda}{2}\sqrt{-\frac{6}{\beta}}-\frac{\alpha}{2\beta}\mp\frac{3\lambda}{\beta}\sqrt{-\frac{\beta}{6}}$$

where $\xi=x-st+\xi_0,\ s=-\dfrac{2\beta\lambda^2+\alpha^2-8\beta\mu}{4\beta}.$

The simple equation method with the Riccati equation

Case 1: $\alpha\beta<0$

$$u(x,t)=-\frac{m}{2n}\pm\frac{m\sqrt{6}}{4n}\tanh\left(\sqrt{-\frac{m^2}{16n}}\left(x+\left(\frac{m^2}{8n}\right)t\right)-\frac{\ln\xi_0}{2}\right)$$

Case 2: $\alpha\beta>0$

$$u(x,t)=\frac{-m}{2n}\pm\frac{m\sqrt{-6}}{4n}\tan\sqrt{\frac{m^2}{16n}}\left(x+\left(\frac{m^2}{8n}\right)t\right)$$

Table 2 Solutions comparison of the combined KdV-mKdV equation between the rational (G′/G)-expansion method and the modified extended tanh function method.

The rational (G′/G)-expansion method

Case 1: $\lambda^2-4\mu>0$

$$u(\xi)=\pm\frac{1}{2}\sqrt{-\frac{6(\lambda^2-4\mu)}{\beta}}\left(\frac{c_1\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi\right)+c_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi\right)}{c_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi\right)+c_2\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi\right)}\right)$$
$$\pm\frac{\lambda}{2}\sqrt{-\frac{6}{\beta}}-\frac{\alpha}{2\beta}\mp\frac{3\lambda}{\beta}\sqrt{-\frac{\beta}{6}}$$

Case 2: $\lambda^2-4\mu<0$

$$u(\xi)=\pm\frac{1}{2}\sqrt{-\frac{6(4\mu-\lambda^2)}{\beta}}\left(\frac{-c_1\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)+c_2\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}{c_1\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)+c_2\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}\right)$$
$$\pm\frac{\lambda}{2}\sqrt{-\frac{6}{\beta}}-\frac{\alpha}{2\beta}\mp\frac{3\lambda}{\beta}\sqrt{-\frac{\beta}{6}}$$

Case 3: $\lambda^2-4\mu=0$

$$u(\xi)=\pm\sqrt{-\frac{6}{\beta}}\left(\frac{c_2}{c_1+c_2\xi}\right)\mp\frac{\lambda}{2}\sqrt{-\frac{6}{\beta}}-\frac{\alpha}{2\beta}\mp\frac{3\lambda}{\beta}\sqrt{-\frac{\beta}{6}}$$

where $\xi=x-st+\xi_0,\ s=-\dfrac{2\beta\lambda^2+\alpha^2-8\beta\mu}{4\beta}.$

The modified extended tanh function method

Case 1: $\sigma<0$

$$u(x,t)=\frac{-m}{2n}\left(1\pm\frac{1}{2}\tanh\sqrt{\frac{-m^2}{96n}}\left(x+\frac{m^2}{6n}t\right)\pm\frac{1}{2\tanh\sqrt{\frac{-m^2}{96n}}\left(x+\frac{m^2}{6n}t\right)}\right)$$

and

$$u(x,t)=\frac{-m}{2n}\left(1\pm\frac{1}{2}\coth\sqrt{\frac{-m^2}{96n}}\left(x+\frac{m^2}{6n}t\right)\pm\frac{1}{2\coth\sqrt{\frac{-m^2}{96n}}\left(x+\frac{m^2}{6n}t\right)}\right)$$

Case 2: $\sigma>0$

$$u(x,t)=\frac{-m}{2n}\left(1\pm\frac{I}{2}\tan\sqrt{\frac{m^2}{96n}}\left(x+\frac{m^2}{6n}t\right)\mp\frac{I}{2\tan\sqrt{\frac{m^2}{96n}}\left(x+\frac{m^2}{6n}t\right)}\right)$$

and

$$u(x,t)=\frac{-m}{2n}\left(1+\frac{I}{2}\cot\sqrt{\frac{m^2}{96n}}\left(x+\frac{m^2}{6n}t\right)-\frac{I}{2\tan\sqrt{\frac{m^2}{96n}}\left(x+\frac{m^2}{6n}t\right)}\right)$$

6. CONCLUSION

In this work, we applied the simple equation method with the Riccati equation and the modified extended tanh function method in a satisfactory way to determine the traveling wave solution for the combined KdV-mKdV equation. Sequently, we found several new exact traveling wave solutions containing trigonometric functions and hyperbolic functions.

Both the simple equation method with the Riccati equation and the modified extended tanh function method rely on the Riccati equation and are straightforward to comprehend. Also, this research shows that this suggested method is suitable and highly practical for finding exact solutions to the combined KdV-mKdV problem. The method provides correct solutions for solitary waves and operates in a dependable and efficient manner.

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