Journal of Applied Science and Emerging Technology (JASET)
Published by Faculty of Applied Science, KMUTNB, Thailand
doi: 10.14416/JASET.KMUTNB.2024.01.005
Vol. 23, No. 1 [2024]: e252913

On Solving the Quantum Oscillator Equation by Using the Differential Transformation

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ABSTRACT

In this paper, we present a simple yet accurate semi-analytical solution of the quantum oscillator equation, which is an eigenvalue problem, by using the differential transformation method (DTM). The quantum oscillator equation is studied under various boundary conditions via differential associated coefficients. Eigenfunctions and eigenvalues are calculated semi-analytically and shown graphically. They are verified and shown to be accurate.

KEYWORDS: Differential Transformation Method, Eigenvalue Problem, Quantum Oscillator Equation.

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Received: 02/06/2023; **Revised:** 27/02/2024; **Accepted:** 01/03/2024

1. INTRODUCTION

The differential transformation method (DTM) is one among numerical methods and can be used as a semi-analytical method to a certain class of ordinary differential equations, partial differential equations and integral equations (Hussin et al. (2010); Hassan (2008); Jang et al. (2001); Ayaz (2004); Ayaz (2014); Zhou (1986)) used the DTM to solve various scientific problems such as linear and non-linear initial value problems in many electric circuit problems. Chen and Ho (1996) solved eigenvalue problems by using the DTM. Later on, Hassan (2002) studied and improved the DTM in order to solve eigenvalue problems more comprehensively.

In this paper, we present a simple yet accurate semi-analytical solution of the quantum oscillator equation using the DTM method. We found that this semi-analytical approach is very simple and easier to write a program to find the ith eigenfuctions compared to other existing analytical and numerical methods. Moreover, we found that the obtained results are in the polynomial form.

The definitions, theorems and some basic mathematical operations of the DTM for solving our problems are stated in the section 2. A procedure involved of how the semi-analytical methods using the DTM is employed to solve an eigenvalue problem of the quantum oscillator equation is shown in section 3. Finally, we illustrate the quantum oscillator equation under some boundary conditions using different associated coefficients and apply the DTM to find a semi-analytical solutions.

The eigenvalue problem of the quantum oscillator equation is shown below

$$y''(x)+(\lambda - x^2)y(x)=0 \tag{1.1}$$

subjected to the following boundary conditions

$$\alpha_1 y(0)-\beta_1 y'(0)=0, \tag{1.2}$$

$$\alpha_2 y(1)+\beta_2 y'(1)=0, \tag{1.3}$$

where $\alpha_i \geq 0,\ \beta_i \geq 0$ and $\alpha_i+\beta_i \geq 0$ for $i=1,2.$

2. MATERIALS AND METHODS

2.1 The differential transformation methods

Definition 2.1. The differential transformation of the function $y(x)$ is defined as follows:

$$Y(k)=\frac{1}{k!}\left[\frac{d^k}{dx^k}y(x)\right]_{x=0} \quad (2.1)$$

where $y(x)$ is the original function and $Y(k)$ is the transformed function.

Definition 2.2. The differential inverse transformation of the function $y(x)$ is defined as follows:

$$y(x)=\sum_{k=0}^{\infty}x^kY(k). \quad (2.2)$$

We have from (2.1) and (2.2) that

$$y(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}\left[\frac{d^k}{dx^k}y(x)\right]_{x=0}. \quad (2.3)$$

Eq. (2.3) implies that the DTM is developed based on the Taylor's series expansion. This method constructs a semi- analytical solution in the form of polynomial.

2.2 Some basic mathematical operations

The following is the basic mathematical operations of the DTM which is necessary for solving our problems. The proof of them can be found in Kerdpol (2015).

Original function	Transformed function	
$y(x)=\lambda\varphi(x)$	$Y(k)=\lambda\Phi(k),$	(2.4)
$y(x)=\varphi(x)\pm\theta(x)$	$Y(k)=\Phi(k)\pm\Theta(k),$	(2.5)
$y(x)=\frac{d\varphi(x)}{dx}$	$Y(k)=(k+1)\Phi(k+1),$	(2.6)
$y(x)=\frac{d^2\varphi(x)}{dx^2}$	$Y(k)=(k+1)(k+2)\Phi(k+2),$	(2.7)
$y(x)=\varphi(x)\theta(x)$	$Y(k)=\sum_{l=0}^{k}\Phi(l)\Theta(k-l),$	(2.8)
$y(x)=x^m$	$Y(k)=\delta(k-m)=\begin{cases}1\,;\,k=m,\\0\,;\,k\neq m,\end{cases}$	(2.9)
$y(x)=x^2\varphi(x)$	$Y(k)=\sum_{l=0}^{k}\delta(l-2)\Phi(k-l),$ where $\delta(l-2)=\begin{cases}1\,;\,l=2\\0\,;\,l\neq 2.\end{cases}$	(2.10)

3. EIGENVALUE PROBLEMS OF QUANTUM OSCILLATOR EQUATION USING DTM

We consider the regular Strum-Liouville eigenvalue problem.

$$\frac{d}{dx}\left[p(x)\frac{dy(x)}{dx}\right]+\left[\lambda r(x)-q(x)\right]y(x)=0 \quad (3.1)$$

with the following boundary conditions

$$\alpha_1y(0)-\beta_1y'(0)=0, \quad (3.2)$$

$$\alpha_2y(1)+\beta_2y'(1)=0, \quad (3.3)$$

where $p(x)>0, r(x)>0$ and $q(x)\geq 0$ and $p(x)$, $r(x)$, $q(x)$ and $p'(x)$ are continuous on the closed interval [0, 1] and $\alpha_i\geq 0, \beta_i\geq 0$ and $\alpha_i+\beta_i\geq 0$ for $i=1,2$.

In case of the quantum oscillator equation, the value of $p(x), r(x)$ and $q(x)$ are $p(x)=1,$ $r(x)=1$ and $q(x)=x^2$ and, as a result, we obtain

$$y''(x)+(\lambda-x^2)y(x)=0. \quad (3.4)$$

Transforming (3.4) by using operations (2.4), (2.5), (2.7), (2.10), we obtain

$$Y(k+2)=\frac{\left(\sum_{l=0}^{k}\delta(l-2)Y(k-l)\right)-\lambda Y(k)}{(k+1)(k+2)}. \quad (3.5)$$

Transforming the boundary conditions (3.2) and (3.3) by using operations (2.1) and (2.2), we obtain

$$\alpha_1Y(0)-\beta_1Y(1)=0, \quad (3.6)$$

$$\sum_{k=0}^{n}(\alpha_2+\beta_2k)Y(k)=0. \quad (3.7)$$

Assume that $Y(0)=c.$ (3.8)

From (3.6) we have

$$Y(1)=\frac{\alpha_1c}{\beta_1}. \quad (3.9)$$

Considering (3.5) when $k=0$ and applying (3.8), (3.9) and the value of function $\delta(l-2)$ from (2.10), we get

$$Y(2)=\frac{-c\lambda}{2}. \quad (3.10)$$

Considering (3.5) when $k=1$ and applying (3.8)-(3.10) and the value of function $\delta(l-2)$ from (2.10), we obtain

$$Y(3)=\frac{-\alpha_1c\lambda}{6\beta_1}. \quad (3.11)$$

Similarly,

$$Y(4)=\frac{(2-\lambda^2)c}{24}. \quad (3.12)$$

The same way is used to find $Y(5)$ until $Y(n)$ and then substituting $Y(1)$ to $Y(n)$ into (3.7) we get

$$\sum_{k=0}^{n}(\alpha_2+\beta_2 k)Y(k)=c\left[f^{(n)}(\lambda)\right]=0, \quad (3.13)$$

where $f^{(n)}(\lambda)$ is a polynomial of λ corresponding to n. In case that $c\neq 0$, we have

$$f^{(n)}(\lambda)=0. \quad (3.14)$$

Solving (3.14), we obtain $\lambda=\lambda_i^{(n)}$, i=1,2,3,.. where $\lambda_i^{(n)}$ is the i th estimated eigenvalue corresponding to n and $\lambda_i^{(n)}$ has to be real root and n satisfies

$$\left|\lambda_i^{(n)}-\lambda_i^{(n-1)}\right|\le\xi \quad (3.15)$$

where $\lambda_i^{(n-1)}$ is the ith estimated eigenvalue corresponding to $n-1$.

Considering (3.15), in this work, we set $\xi=0.01$ then the following procedure is used for finding $\lambda_i^{(n)}$.

In case that (3.15) is satisfied, we will choose $\lambda_i^{(n)}$ to be the ith eigenvalue corresponding to n and then substituting $\lambda_i^{(n)}$ into $Y(0)$ to $Y(n)$ and using (2.2), we get

$$y_i(x)=\sum_{k=0}^{n}x^k Y_{\lambda_i^n}(k) \quad (3.16)$$

where $y_i(x)$ is ith eigenfunction corresponding to the eigenvalue λ_i.

Note that the function $y_i(x)$ in (3.16) is expressed by a finite series of $\sum_{k=n+1}^{\infty}x^kY(k)$ which is negligibly small and n is decided by the convergence of the eigenvalue.

The i th normalized eigenfunction is defined as follows:

$$\hat{y}_i(x)=\frac{y_i(x)}{\int_0^1|y_i(x)|dx}. \quad (3.17)$$

In case that (3.15) is unsatisfied, we will replace n by $n+1$ and repeat the same procedure in (3.13)-(3.15) until (3.15) is satisfied.

4. SOLVING PROBLEM AND MAIN RESULT

In this section, we will show how to apply the DTM to solve the quantum oscillator equation under various boundary conditions via differential associated coefficients. The obtained solutions are substituted back into the boundary eigenvalue problems to verify our proposed method accuracy.

Problem 4.1.

$$y''(x)+(\lambda-x^2)y(x)=0, \quad (4.1)$$

$$y'(0)=0, \quad (4.2)$$

$$y(1)=0. \quad (4.3)$$

From transformation (3.5), we have

$$Y(k+2)=\frac{\left(\sum_{l=0}^{k}\delta(l-2)Y(k-l)\right)-\lambda Y(k)}{(k+1)(k+2)}. \quad (4.4)$$

From transformation (3.6) and (3.7) when $\alpha_1=0$, $\beta_1=1, \alpha_2=1$ and $\alpha_2=0$ we have

$$Y(1)=0, \quad (4.5)$$

$$\sum_{k=0}^{n}Y(k)=0. \quad (4.6)$$

(I) *Solving the first eigenvalue and eigenfunction:*

Assume that $Y(0)=c$. (4.7)

Considering (4.4) when $k=0$ and applying (4.5), (4.7) and the value of function $\delta(l-2)$ from (2.10), we obtain

$$Y(2)=\frac{-c\lambda}{2}. \quad (4.8)$$

Considering (4.4) when $k=1$ and applying (4.7) - (4.8) and the value of function $\delta(l-2)$ from (2.10), we obtain $Y(3)=0.$ (4.9)

Similarly, $Y(4), Y(5)$ can be founded by substituting $k=2$ and $k=3$ into (4.4), respectively. We obtain that

$$Y(4)=\frac{c(2+\lambda^2)}{24},$$

$$Y(5)=0.$$

In our work we set n=10 and using the same way to find $Y(6)$ until $Y(10)$

$$Y(6)=-\frac{c\lambda(14+\lambda^2)}{720},$$

$$Y(8)=\frac{c(60+44\lambda^2+\lambda^4)}{40320},$$
$$Y(10)=-\frac{c\lambda(844+100\lambda^2+\lambda^4)}{3628800}.$$

At $Y(2k+1)=0$ for $k=1,2,3,\ldots$

Substituting $Y(0)$ until $Y(10)$ into (4.6), we obtain a polynomial of λ corresponding to

$$\sum_{k=0}^{10}Y(k)=c[f^{(10)}(\lambda)]=0,$$

$$f^{(10)}(\lambda)=1-\frac{\lambda}{2}+\frac{2+\lambda^2}{24}-\frac{\lambda(14+\lambda^2)}{720}+\frac{60+44\lambda^2+\lambda^4}{40320}-\frac{\lambda(844+100\lambda^2+\lambda^4)}{3628800}=0. \quad (4.10)$$

Solving (4.10), we get

$$\lambda=2.597,\ 19.081\pm45.154i,\ 24.621\pm4.972i.$$

Choosing real root $\lambda_1^{(10)}=2.597.$ (4.11)

When $n=8$, substituting $Y(0)$ to $Y(8)$ into (4.6), we obtain

$$\sum_{k=0}^{8}Y(k)=c[f^{(8)}(\lambda)]=0.$$

$$f^{(8)}(\lambda)=1-\frac{\lambda}{2}+\frac{2+\lambda^2}{24}-\frac{\lambda(14+\lambda^2)}{720}+\frac{60+44\lambda^2+\lambda^4}{40320}=0.$$

Solving the above equation, we have

$$\lambda=2.60,\ 17.621,\ 17.89\pm25.193i.$$

Choosing $\lambda_1^{(8)}=2.60.$ (4.12)

Substituting (4.11) and (4.12) into (3.15) we obtain

$$\left|\lambda_1^{(10)}-\lambda_1^{(8)}\right|=\left|2.597-2.60\right|=0.003\le\xi. \quad (4.13)$$

Since (4.13) is satisfied, so we choose $\lambda_1^{(10)}=2.597$ to be the first eigenvalue and substituting λ_1 into $Y(0)$ to $Y(10)$ and by (3.16), we get the first eigenfunction.

$$\begin{aligned}y_1(x)&=\sum_{k=0}^{n}x^kY(k)\\&=Y(0)+x^2Y(2)+x^4Y(4)+x^6Y(6)+x^8Y(8)+x^{10}Y(10)\\&=c-\frac{c\lambda}{2}x^2+\frac{c(2+\lambda^2)}{24}x^4-\frac{c\lambda(14+\lambda^2)}{720}x^6+\frac{c(60+44\lambda^2+\lambda^4)}{40320}x^8\\&\quad-\frac{c\lambda(844+100\lambda^2+\lambda^4)}{3628800}x^{10}\\&=\begin{pmatrix}1-1.3x^2+0.365x^4-0.0749667x^6\\+9.99845\times10^{-3}x^8-1.12181\times10^{-3}x^{10}\end{pmatrix}c.\end{aligned}$$

By using (3.17), the first normalized eigenfunction is shown below.

$$\begin{aligned}\hat{y}_1(x)&=\frac{y_i(x)}{\int_0^1\left|y_i(x)\right|dx}\\&=\frac{\begin{pmatrix}1-1.3x^2+0.365x^4-0.0749667x^6\\+9.99845\times10^{-3}x^8-1.12181\times10^{-3}x^{10}\end{pmatrix}c}{\left.\begin{pmatrix}x-\frac{1.3x^3}{3}+\frac{0.365x^5}{5}-\frac{0.0749667x^7}{7}\\+\frac{9.99845\times10^{-3}x^9}{9}-\frac{1.12181\times10^{-3}x^{19}}{19}\end{pmatrix}\right|_0^1c}\\&=1.58739\begin{pmatrix}1-1.3x^2+0.365x^4-0.0749667x^6\\+9.99845\times10^{-3}x^8-1.12181\times10^{-3}x^{10}\end{pmatrix}.\end{aligned} \quad (4.14)$$

Finally, we plot the calculated normalized first eigenfunction in (4.14) as shown in Fig. 1. In addition, by substituting (4.14) into the quantum oscillator equation and boundary conditions in (4.1)-(4.3), satisfactory results are achieved where the corresponding errors are very close to zero and summarized in Table 1.

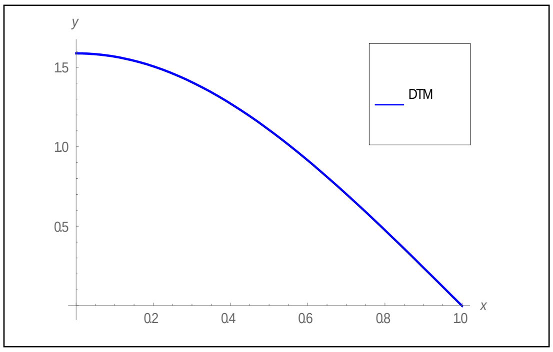

Figure 1 The calculated first eigenfunction (4.14) by DTM.

Table 1 Corresponding errors after substituting the DTM first eigenfunction (4.14) to the problem.

x	$\hat{y}_1''(x)+(\lambda-x^2)\hat{y}_1(x)$
0	0
0.1	2.04834*10^-12
0.2	2.09204*10^-9
0.3	1.20112*10^-7
0.4	2.11985*10^-6
0.5	0.0000195861
0.6	-0.000120087
0.7	-0.000554464
0.8	-0.00207894
0.9	-0.00664544
1	-0.0187206

(II) *Solving the second eigenvalue and eigenfunction:*

When $n=18$ and using the same procedure, we solve $f^{(18)}(\lambda)=0$ and choose the real roots, we have $\lambda_1^{(18)}=2.597$ and $\lambda_2^{(18)}=22.52.$

Since $\lambda_1^{(18)} = \lambda_1^{(10)} = 2.597$ then we use $\lambda_2^{(18)} = 22.52$ and $\left|\lambda_2^{(18)} - \lambda_2^{(16)}\right| = |22.52 - 22.51| = 0.01 \le \xi$, so we choose $\lambda_2^{(18)} = 22.52$ to be the second eigenvalue. Substituting λ_1 into $Y(0)$ to $Y(18)$ and using (3.17), the second normalized eigenfunction is calculated as

$$\hat{y}_2(x) = -4.60671\begin{pmatrix} 1 - 11.26x^2 + 21.2146x^4 - 16.3004x^6 + 6.93393x^8 \\ -1.91614x^{10} + 0.379435x^{12} - 0.0574781x^{14} \\ +0.00697434x^{16} - 7.01112\times10^{-4}x^{18} \end{pmatrix}. \tag{4.15}$$

Similarly, we plot the calculated normalized second eigenfunction in (4.15) as shown in Figure 2 and after substituting (4.15) into the quantum oscillator equation and boundary conditions in (4.1)-(4.3), we achieve satisfactory results. The corresponding errors are again close to zero and listed in Table 2.

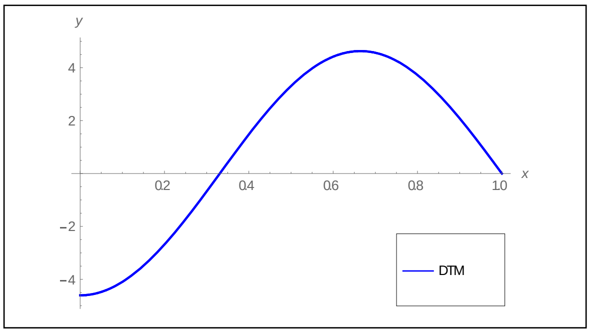

Figure 2 The calculated second eigenfunction (4.15) by DTM.

Table 2 Corresponding errors after substituting the DTM second eigenfunction (4.15) to the problem.

x	$\hat{y}_2''(x) + (\lambda - x^2)\hat{y}_2(x)$
0	0
0.1	4.53511*10^-17
0.2	2.81684*10^-14
0.3	4.05175*10^-11
0.4	7.17072*10^-9
0.5	3.96946*10^-7
0.6	0.0000105319
0.7	0.000168185
0.8	0.00185183
0.9	0.0153469
1	0.101635

(III) *Solving the third eigenvalue and eigenfunction:*

When $n = 28$ and apply the same procedure, we solve $f^{(28)}(\lambda) = 0$ and choose real roots, we have

$\lambda_1^{(28)} = 2.60$, $\lambda_2^{(28)} = 22.52$ and $\lambda_3^{(28)} = 62.01$. Since $\lambda_1^{(28)} = \lambda_1^{(18)} = \lambda_1^{(10)} = 2.60$ and $\lambda_2^{(28)} = \lambda_2^{(18)} = 22.52$ then we use $\lambda_3^{(28)} = 62.01$ from $\left|\lambda_3^{(28)} - \lambda_3^{(26)}\right| \le \xi$, so we choose $\lambda_3^{(28)} = 62.01$ to be the third eigenvalue and then the third normalized eigenfunction is calculated as

$$\hat{y}_3(x) = 7.78311\begin{pmatrix} 1 - 31.005x^2 + 160.302x^4 - 332.377x^6 + 370.911x^8 - 259.251x^{10} \\ +124.599x^{12} - 43.877x^{14} + 11.8559x^{16} - 2.54595x^{18} \\ +0.446659x^{20} - 0.0654616x^{22} + 0.00816292x^{24} \\ -8.79453\times10^{-4}x^{26} + 8.29336\times10^{-5}x^{28} \end{pmatrix}. \tag{4.16}$$

We plot the calculated normalized third eigenfunction in (4.16) as shown in Figure 3. Similarly, after substituting (4.16) into the quantum oscillator equation and boundary conditions in (4.1)-(4.3), we achieve satisfactory results. The corresponding errors are close to zero and listed in Table 3.

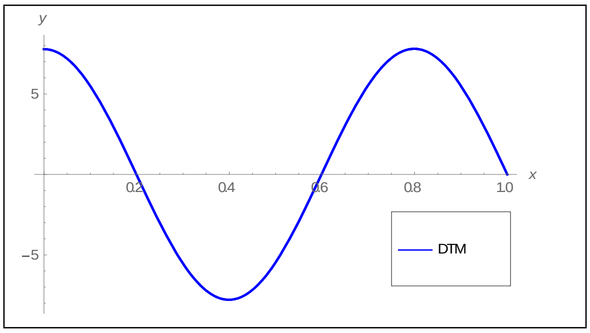

Figure 3 The calculated third eigenfunction (4.16) by DTM.

Table 3 Corresponding errors after substituting the DTM third eigenfunction (4.16) to the problem.

x	$\hat{y}_3''(x) + (\lambda - x^2)\hat{y}_3(x)$
0	0.
0.1	-3.78638*10^-14
0.2	-1.7059*10^-13
0.3	-4.64489*10^-13
0.4	-7.17753*10^-13
0.5	1.71847*10^-10
0.6	2.86365*10^-8
0.7	2.14145*10^-6
0.8	0.0000898629
0.9	0.00242563
1	0.0462257

Problem 4.2.

$$y''(x) + (\lambda - x^2)y(x) = 0, \tag{4.17}$$

$$2y(0)-y'(0)=0, \quad (4.18)$$

$$3y(1)+y'(1)=0. \quad (4.19)$$

Taking differential transformation of (4.17), we obtain

$$Y(k+2)=\frac{\left(\sum_{l=0}^{k}\delta(l-2)Y(k-l)\right)-\lambda Y(k)}{(k+1)(k+2)}. \quad (4.20)$$

Using (2.1), the boundary condition (4.18) becomes

$$2Y(0)-Y(1)=0. \quad (4.21)$$

Using (2.2), the boundary condition (4.19) becomes

$$\sum_{k=0}^{n}(3+k)Y(k)=0. \quad (4.22)$$

(I) *Solving the first eigenvalue and eigenfunction*

Let $Y(0)=c.$ (4.23)

Substituting (4.23) into (4.21), we have

$$Y(1)=2c. \quad (4.24)$$

At $k=0$ and substituting (4.23), (4.24) and the function $\delta(l-2)$ from (2.10) into (4.20), we have

$$Y(2)=\frac{-c\lambda}{2}. \quad (4.25)$$

At $k=1$ and substituting from (4.23) to (4.25) and the function $\delta(l-2)$ from (2.10) into (4.20), we have $Y(3)=\frac{-c\lambda}{3}.$ (4.26)

At $k=2$ and substituting from (4.23) to (4.26) and the function $\delta(l-2)$ from (2.10) into (4.20), we have $Y(4)=\frac{c(2+\lambda^2)}{24}.$ (4.27)

At $k=3$ and substituting from (4.23) to (4.27) and the function $\delta(l-2)$ from (2.10) into (4.20), we have

$$Y(5)=\frac{c(6+\lambda^2)}{60}. \quad (4.28)$$

From the above procedure, we found the result corresponding to $n=11$ as

$$Y(6)=-\frac{c\lambda(14+\lambda^2)}{720},$$

$$Y(7)=-\frac{c\lambda(26+\lambda^2)}{2520},$$

$$Y(8)=\frac{c(60+44\lambda^2+\lambda^4)}{40320},$$

$$Y(9)=\frac{c(844+100\lambda^2+\lambda^4)}{3628800},$$

$$Y(10)=-\frac{c\lambda(844+100\lambda^2+\lambda^4)}{3628800},$$

$$Y(11)=-\frac{c\lambda(2124+140\lambda^2+\lambda^4)}{19958400}. \quad (4.29)$$

Substituting from $Y(0)$ to $Y(11)$ into (4.22), we obtain

$$\sum_{k=0}^{11}Y(k)=c[f^{(11)}(\lambda)]=0,$$

$$f^{(11)}(\lambda)=11-\frac{5\lambda}{2}-2\lambda+\frac{7(2+\lambda^2)}{24}+\frac{2(6+\lambda^2)}{15} -\frac{\lambda(14+\lambda^2)}{80}-\frac{\lambda(26+\lambda^2)}{252}+\frac{11(60+44\lambda^2+\lambda^4)}{40320} +\frac{252+68\lambda^2+\lambda^4}{15120}-\frac{13\lambda(844+100\lambda^2+\lambda^4)}{3628800} -\frac{\lambda(2124+140\lambda^2+\lambda^4)}{1425600}=0. \quad (4.30)$$

Solving (4.30), we get

$$\lambda=3.685,\ 17.323\pm39.109i,\ 20.3956\pm3.7289i.$$

Choose $\lambda_1^{(10)}=3.685.$ (4.31)

At $n=10$ Substituting from $Y(0)$ to $Y(11)$ into (4.22) we obtain

$$\sum_{k=0}^{10}Y(k)=c[f^{(10)}(\lambda)]=0,$$

$$f^{(10)}(\lambda)=11-\frac{5\lambda}{2}-2\lambda+\frac{7(2+\lambda^2)}{24}+\frac{2(6+\lambda^2)}{15} -\frac{\lambda(14+\lambda^2)}{80}-\frac{\lambda(26+\lambda^2)}{252}+\frac{11(60+44\lambda^2+\lambda^4)}{40320} +\frac{252+68\lambda^2+\lambda^4}{15120}-\frac{13\lambda(844+100\lambda^2+\lambda^4)}{3628800}=0.$$

Solving the above equation we have

$$\lambda=3.69,\ 17.8975,\ 29.6899,\ 21.6698\pm36.0299i.$$

Choose $\lambda_1^{(10)}=3.69.$ (4.32)

By substituting (4.31) and (4.32) into (3.15) we obtain

$$\left|\lambda_1^{(11)}-\lambda_1^{(10)}\right|=|3.685-3.69|=0.005\le\xi. \quad (4.33)$$

From (4.33), $\lambda_1=3.69$ is the first eigenvalue. Substituting λ_1 into $Y(0)$ to $Y(11)$ and using (3.16) we have the first eigenfunction.

$$\begin{aligned} y_1(x)&=\sum_{k=0}^{11}x^kY(k)\\ &=Y(0)+xY(1)+x^2Y(2)+\ldots+x^{10}Y(10)\\ &=c+2cx-\frac{c\lambda}{2}x^2-\frac{c\lambda}{3}x^3+\frac{c(2+\lambda^2)}{24}x^4+\frac{c(6+\lambda^2)}{60}x^5\\ &\quad-\frac{c\lambda(14+\lambda^2)}{720}x^6-\frac{c\lambda(26+\lambda^2)}{2520}x^7+\frac{c(60+44\lambda^2+\lambda^4)}{40320}x^8\\ &\quad+\frac{c(252+68\lambda^2+\lambda^4)}{181440}x^9-\frac{c\lambda(844+100\lambda^2+\lambda^4)}{3628800}x^{10}\\ &\quad-\frac{c\lambda(2124+140\lambda^2+\lambda^4)}{19958400}x^{11} \end{aligned}$$

$$= \begin{pmatrix} 1+2x-1.845x^2-1.23x^3+0.650671x^4+0.326935x^5 \\ -0.141533x^6-0.0580093x^7+0.0209451x^8 \\ +0.00751374x^9-0.00243133x^{10}-7.79409\times 10^{-4}x^{11} \end{pmatrix} c.$$

Using (3.17), the first normalized eigenfunction is shown belows

$$\hat{y}_1(x) = \frac{y_i(x)}{\int_0^1 |y_i(x)|dx}$$

$$= \frac{\begin{pmatrix} 1+2x-1.845x^2-1.23x^3+0.650671x^4+0.326935x^5 \\ -0.141533x^6-0.0580093x^7+0.0209451x^8 \\ +0.00751374x^9-0.00243133x^{10}-7.79409\times 10^{-4}x^{11} \end{pmatrix} c}{\left.\begin{pmatrix} x+x^2-0.615x^3-0.3075x^4+0.130134x^5+0.05449x^6 \\ -0.0202189x^7-0.00725116x^8+0.00232723x^9 \\ +0.000751374x^{10}-0.00022103x^{11}-6.49507\times 10^{-5}x^{12} \end{pmatrix}\right|_0^1 c}$$

$$= 0.808116 \begin{pmatrix} 1+2x-1.845x^2-1.23x^3+0.650671x^4+0.326935x^5 \\ -0.141533x^6-0.0580093x^7+0.0209451x^8 \\ +0.00751374x^9-0.00243133x^{10}-7.79409\times 10^{-4}x^{11} \end{pmatrix}. \quad (4.34)$$

The calculated solutions from (4.34) is shown in Figure 4.

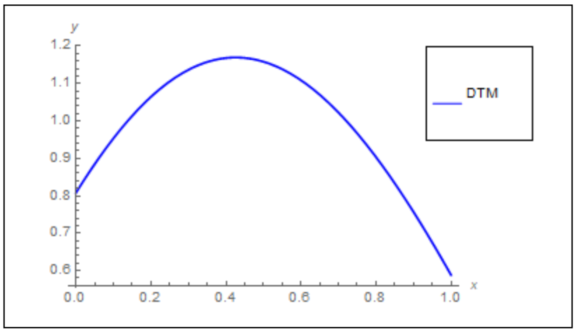

Figure 4 The calculated first eigenfunction (4.34) by DTM.

Table 4 The values of the problem close to zero after substituting the DTM first eigenfunction (4.34) to the problem.

x	$\hat{y}_1''(x)+(\lambda-x^2)\hat{y}_1(x)$
0	0
0.1	2.04834*10^-12
0.2	2.09204*10^-9
0.3	1.20112*10^-7
0.4	2.11985*10^-6
0.5	0.0000195861
0.6	-0.000120087
0.7	-0.000554464
0.8	-0.00207894
0.9	-0.00664544
1	-0.0187206

(II) *Solving the second eigenvalue and eigenfunction*

When $n=18$ and apply the same procedure, we solve $f^{(18)}(\lambda)=0$ and choose real roots, we have

$\lambda_1^{(18)} = 3.69,\ \lambda_2^{(18)} = 18.06.$

Note that since $\lambda_1^{(18)} = \lambda_1^{(11)}$ and $\left|\lambda_2^{(18)} - \lambda_2^{(17)}\right| = |18.05-18.06| = 0.01 \le \xi$ so we get the second eigenvalue $\lambda_2 = 18.06$ and the second normalized eigenfunction is shown as

$$\hat{y}_2(x) = -18.6613 \begin{pmatrix} 1+2x-9.03x^2-6.02x^3+13.6735x^4+5.53606x^5-8.53244x^6 \\ -2.52384x^7+2.99588x^8+0.709953x^9-0.695978x^{10}-0.13951x^{11} \\ +0.117919x^{12}+0.0207014x^{13}-0.0155252x^{14}-0.00244463x^{15} \\ +0.0016596x^{16}+2.38424\times 10^{-4}x^{17}-148685\times 10^{-4}x^{18} \end{pmatrix}. \quad (4.35)$$

The calculated normalized semi-analytical solutions from (4.35) is shown in Figure 5.

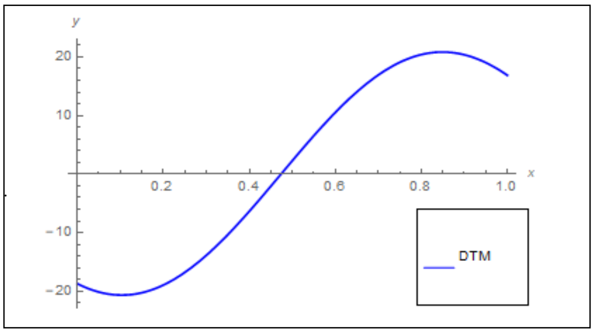

Figure 5 The calculated second eigenfunction (4.35) by DTM.

Table 5 The values of the problem close to zero after substituting the DTM second eigenfunction (4.35) to the problem.

x	$\hat{y}_2''(x)+(\lambda-x^2)\hat{y}_2(x)$
0	0
0.1	9.0226*10^-17
0.2	-1.42174*10^-13
0.3	-1.30843*10^-10
0.4	-1.59786*10^-8
0.5	-6.45971*10^-7
0.6	-0.0000129191
0.7	-0.000158165
0.8	-0.00134395
0.9	-0.00857555
1	-0.0432194

(III) *Solving the third eigenvalue and eigenfunction:*

When $n=25$, and using the same procedure, we solve $f^{(25)}(\lambda)=0$ and choose real roots, we have

$\lambda_1^{(25)} = 3.69,$

$\lambda_2^{(25)} = 18.06,$

$\lambda_3^{(25)} = 48.93.$

Note because $\lambda_1^{(25)} = \lambda_1^{(18)} = \lambda_1^{(11)}$ and $\lambda_2^{(25)} = \lambda_2^{(18)}$ from $\left|\lambda_3^{(25)} - \lambda_3^{(24)}\right| \le \xi$, we get the second eigenvalue $\lambda_3 = 48.93$ and the second normalized eigenfunction is shown as

$$\hat{y}_3(x) = 9.83913\begin{pmatrix} 1+2x-24.465x^2-16.31x^3+99.8394x^4+40.0024x^5-163.654x^6 \\ -46.9911x^7+144.775x^8+32.49x^9-80.5278x^{10}-14.8793x^{11} \\ +30.947x^{12}+4.87523x^{13}-8.76243x^{14}-1.20678x^{15}+1.91539x^{16} \\ +0.235011x^{17}-0.334909x^{18}-0.0371517x^{19}+0.0481645x^{20} \\ +0.00488773x^{21}-0.00582597x^{22}-5.46063\times10^{-4}x^{23} \\ +6.03676\times10^{-4}x^{24}+5.26777\times10^{-5}x^{25} \end{pmatrix}. \quad (4.36)$$

The calculated solutions from (4.36) is shown in Figure 6.

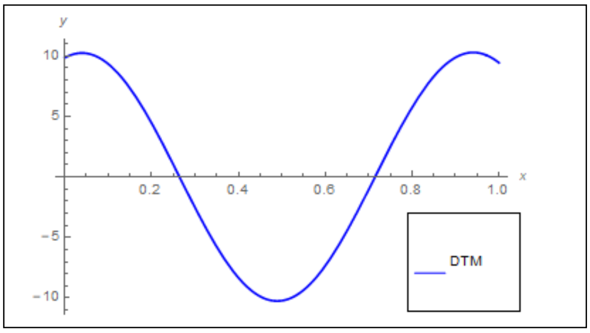

Figure 6 The calculated third eigenfunction (4.36) by DTM.

Table 6 The values of the problem close to zero after substituting the DTM third eigenfunction (4.36) to the problem.

x	$\hat{y}_3''(x)+(\lambda-x^2)\hat{y}_3(x)$
0	0
0.1	-6.04217*10^-15
0.2	-3.61957*10^-14
0.3	5.12551*10^-14
0.4	1.01176*10^-10
0.5	2.15634*10^-8
0.6	1.72543*10^-6
0.7	0.0000701905
0.8	0.00174005
0.9	0.0295471
1	0.372225

The eigenvalues of the problems 4.1 and 4.2 are shown in Figure 7.

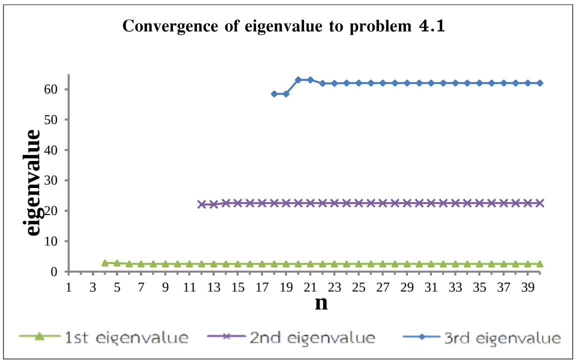

Figure 7 Convergence of eigenvalues λ_1 to λ_3, where λ_1, λ_2 and λ_3 converge to 2.60, 22.52 and 62.01, respectively.

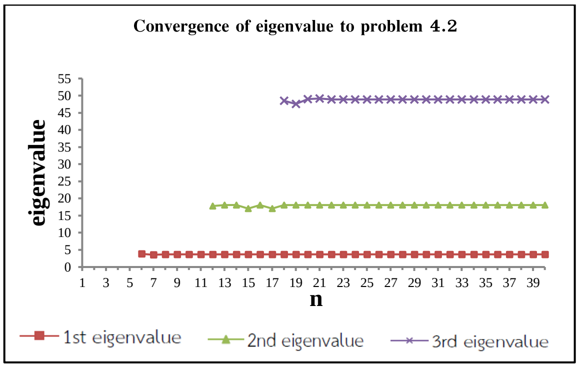

Figure 8 Convergence of eigenvalues λ_1 to λ_3, where λ_1, λ_2 and λ_3 converge to 3.69, 18.06 and 48.93, respectively.

5. CONCLUSION

The differential transformation method is an efficient method which can be used to find the semi-analytical solution of the quantum oscillating equation. The procedure is very straightforward and easy to program. The obtained solutions are shown to be accurate since the corresponding errors are very close to zero after substituting eigenfunctions into the quantum oscillator equation and boundary conditions.

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