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Research Article

An approximation of average run length using numerical integration methods on EWMA control chart for MAX(q,r) process

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Abstract

This research aims to measure a method for estimating the average run length (ARL) of an exponentially weighted moving average control chart using numerical integration when the data is a moving average model with exogenous variables. We compare the average run lengths achieved using three distinct methodologies. The midpoint, trapezoid, and Gaussian rules are all used. Additionally, we compare the CPU time used by the ARL assessment. The simulations show that the ARL values obtained from using the midpoint and Gaussian rules are similar. The result obtained from using the trapezoid approach, on the other hand, is less different at roughly 1%. Additionally, the midpoint and trapezoid approaches were the fastest when CPU time was considered, requiring between 4-6 seconds. On the other hand, Gaussian's rule requires around 37-43 seconds.

Keywords: EWMA, control chart, average run length, numerical integral, moving average

Introduction

Quality control is an essential tool for reducing the many deficiencies throughout the manufacturing process. Statistical Process Control (SPC) is used to design and enhance processes. Control charts are a robust process of monitoring or regulating manufacturing processes in real-time. In 1924, Walter A. Shewhart invented the control chart to help manufacturers minimize waste and enhance quality. Control charts have since been extensively utilized for identifying and monitoring effects on the quality of processes in various applications, including industrial production, public health, computer networking and telecommunications, finance and economics, environmental research, and others. Nowadays, Shewhart control charts, cumulative sum charts (CUSUM) and exponentially weighted moving average control charts (EWMA) are popularly used in industrial processes. Robert (1959) developed the EWMA control chart to detect a slight change in the mean, suitable for normal distribution. It is now well accepted that a strong control chart is essential for identifying small and moderate changes in the process. Control chart performance is often measured by the average run length (ARL). In most cases, the ARL_0 indicates that the process is under control, whereas the ARL_1 suggests that the process is out of control and should be small. The ARL of EWMA control

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chart may be estimated using the Monte Carlo Simulations (MC) technique, which is the conventional way to test the validity and compare it to the other methods. However, processing takes a long time, the Martingale (Sukparangsri & Navikov, 2008), Markov chain (Brook and Evan, 1972; Lucus & Saccicci, 1990) and integral equation (See Srivastava & Wu, 1997; Areepong, 2009; Mittitelu et al.,2010; Petcharat et al., 2013; Paichit, 2017; Preerajit et al. ,2018).

The serially correlated data may occur in the real world of the data set when the data are AR(1) and ARMA(1,1) processes. Lu & Reynolds (1999) employed the integral equation technique to derive the ARL. Some literlature try to evaluate ARL when data set are time series model. Petcharat et al. (2013) proposed the analytical expression for the ARL of the EWMA control chart for the moving average of order q (MA(q))process. For the seasonal $AR(P)_L$ process. Later, Petcharat (2016) derived the explicit formula of ARL for the EWMA chart and compared it to the CUSUM control chart and found that the EWMA chart was more sensitive to detecting small shifts. Paichit (2017) proved the explicit formula of ARL for the EWMA control chart base on autoregressive with an explanatory variable model. Sunthornwa et al. (2018) proved exact solution and approximate the numerical of ARL for EWMA control charts using the seasonal ARFIMA with exponential white noise. Moreover, Suriyakart (2020) studied the sensitivity of the EWMA control chart for detecting the mean change in processes when processes are AR(1), ARMA(1,1) and IMA(1) with exponential white noise and found that the EWMA control chart is good performance for detecting the small change of the process mean. Sunthornwat & Areepong (2020) derived analytical ARL and approximately the numerical ARL for the CUSUM control chart for seasonal and non-seasonal moving averages processes. Recently, Petcharat (2021) presented the exact formula of ARL for the MAX(1,r) process with exponential white noise, with its existence and uniqueness being proved by Banach's fixed-point theory.

According to this literature review, the ARL for measuring the efficiency of the EWMA control chart is very important for comparing control chart performance. In this research, we extend Petcharat (2021) to evaluate the ARL of the EWMA control chart by using numerical integration methods for data is the model moving average model with exogenous variables (MAX(q,r)).

Methods

Exponentially weighted moving average (EWMA) control chart

Robert (1959) initially suggested the EWMA control chart. The recursive equation below would be used to express the EWMA control chart.

$$Y_t = (1-\lambda)Y_{t-1} + \lambda Z_t \ , \quad \mathrm{t} = 1, 2, \ldots, \tag{1}$$

where λ is exponential smoothing parameter, $0 < \lambda < 1$. Z_t is sequence of process observations. The control limit of control charts are upper control limit (UCL) and lower control limit (LCL) as follows,

$$LCL/UCL = \mu \pm L\sigma\sqrt{\frac{\lambda}{2-\lambda}} \ , \tag{2}$$

where μ is average or target value, L is the width of the control limit and σ is standard deviation of the process.

In this research, we present EWMA control chart for the process observations is the sequence of a moving average process with explanatory variables (MAX(q,r)) with exponential white noise defined as

$$Z_t = \mu + \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{i=1}^{r}\beta_i X_{it}, \quad (3)$$

where ε_t is exponential white noise : $\varepsilon_t \sim Exp(\alpha)$, θ_i is moving average coefficient, $i=1,2,..q$, $-1 \le \theta_i \le 1$, X_{it} is exogenous variable and β_i is a coefficient of Xit. $\varepsilon_t \sim \sum_{i=1}^{r}\beta_i X_{it}$

The corresponding stopping time for (1) define as

$$\tau = \inf\{t > 0; Y_t > b\} \quad , H_0 = u, \ \ b > y. \quad (4)$$

where b denote control limit.

Let $\mathbf{E}_\omega(.)$ denote the expectation under probability density function $f(y,\alpha)$ that the change-point occurs at point ω, where $\omega < \infty$. Thus by definition, the ARL for MAX(q,r) process with an initial value $H_0 = u$ is as follow

$$ARL = H(u) = \mathbf{E}_\infty(\tau_b) < \infty. \quad (5)$$

Numerical integral equation (NIE) of average run length for MAX(q,r) on EWMA control chart

Let $H(u)$ be the average run length for the EWMA control chart as defined in eqauation (5). Assume that $C_0 = u$ be the initial process under in-control. The integral equation $H(u)$ can be derived by Fredholm integral equation of the second kind denotes as follows:

$$H(u) = 1 + \frac{1}{\lambda}\int_0^b h(y)\, f\left(\frac{y-(1-\lambda)u}{\lambda} + \mu + \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{i=1}^{r}\beta_i X_{it}\right) d(y),$$

Such that

$$H(u) = 1 + \frac{1}{\lambda\alpha}\int_0^b h(y) e^{\frac{y}{\lambda\alpha}} e^{\frac{(1-\lambda)u}{\lambda\alpha} + \frac{\left(\mu+\varepsilon_t-\theta_1\varepsilon_{t-1}-\theta_2\varepsilon_{t-2}-\ldots-\theta_q\varepsilon_{t-q}+\sum_{i=1}^{r}\beta_i X_{it}\right)}{\alpha}} d(\mathrm{y}), \quad (6)$$

The integral equation $H(u)$ may be approximated using numerical quadrature procedures. The Gaussian rule, the midpoint rule, and the trapezoidal rule are all employed in this research.

I. Gaussian Rule

The quadrature rule is used to approximate an integral, and the Gauss–Legendre quadrature rule is used to approximate the solution. Suppose the $\{\varpi_j, j = 1,\ldots,m\}$ is a point on

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the interval [0,b] and set $\{k_j, j=1,...,m\}$ as $k_j = b/m \geq 0$; $j=1,2,...,m$. The quadrature rule evaluates an integral's approximation as follows:

$$\int_0^b K(y) f(y) dy \approx \sum_{j=1}^{m} \varpi_j f(a_j),$$

where $\varpi_j = \frac{b}{m}\left(j - \frac{1}{2}\right)$; $j = 1,2,...,m$.

It can be written $R_{m\times m}$ be matrix form as

$$\left[R\right]_{jl} = \frac{1}{\lambda}\sum_{j=1}^{m} k_j f\left(\frac{\varpi_k - (1-\lambda)\varpi_m}{\lambda} + \left(\mu + \varepsilon_t - \theta_1\varepsilon_{t-1} - ... - \theta_q\varepsilon_{t-q} + \sum_{i=1}^{r}\beta_i X_{it}\right)\right),\ j,l = 1,2,...,m.$$

Let $H_G(u)$ be the numerical approximation for an integral equation solution in equation (6) which can be found as the linear equations as follows:

$$H_G(u) = 1 + \frac{1}{\lambda}\sum_{j=1}^{m} k_j H(\varpi_k) f(\mu + \varepsilon_t - \theta_1\varepsilon_{t-1} - ... - \theta_q\varepsilon_{t-t}\sum_{i=1}^{r}\beta_i X_{it}), \quad (7)$$

where $\varpi_j = \frac{b}{m}\left(j - \frac{1}{2}\right)$; $j = 1,2,...,m$.

II. Midpoint Rule

By using the midpoint rule**,** Let m be subinterval on [0,b] and let $f(A_j) = f\left(\frac{\varpi_k - (1-\lambda)u}{\lambda} + \left(\mu + \varepsilon_t - \theta_1\varepsilon_{t-1} - ... - \theta_q\varepsilon_{t-q} + \sum_{i=1}^{r}\beta_i X_{it})\right)\right)$.

The approximation for the integral equation solution in equation (6) is given by

$$H_M(u) = 1 + \frac{1}{\lambda}\sum_{j=1}^{m} k_j L(\varpi_j) f(A_j), \quad (8)$$

where $\varpi_j = \frac{b}{m}\left(j - \frac{1}{2}\right)$ and $k_j = \frac{b}{m}$; $j = 1,2,...,m$.

III. Trapezoidal rule

By using the trapezoidal rule, suppose m be subinterval on [0,b]. let $f(A_j) = f\left(\frac{\varpi_k - (1-\lambda)u}{\lambda} + \left(\mu + \varepsilon_t - \theta_1\varepsilon_{t-1} - ... - \theta_q\varepsilon_{t-q} + \sum_{i=1}^{r}\beta_i X_{it})\right)\right)$. The approximation for the integral equation solution in equation (6) is given by

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$$H_T(u) = 1 + \frac{1}{\lambda}\sum_{j=1}^{m+1} k_j L(\varpi_j) f(A_j), \quad (9)$$

where $\varpi_j = jk_j$ and $k_j = \frac{b}{m}$; $j = 1, 2, ..., m-1$, in other case, $k_j = \frac{b}{2m}$.

Results and discussion

In this part, we compare ARL values for EWMA control chart on MAX(q,r) process with exponential white noise obtain from NIE methods which are Gaussian Rule, midpoint rule and Trapezoidal rule according to equation (7) to (9) with m=800 subintervals. We set $H_G(u)$ is ARL from NIE method using the Gaussian rule from explicit formula, $H_M(u)$ is ARL from NIE method using the midpoint rule and $H_T(u)$ is ARL from NIE method using the trapezoidal rule. Additionally, we compare the computing times of three approaches. The computational time for the three methods is estimated in seconds by the central processing unit (CPU) time (Windows 8 OEM, Intel(R) core(TM)i5-8265U CPU@1.60GHz 1.80GHz RAM 8.00 GB (7.89 GB usable)).

In Table 1, the parameters value b for EWMA control chart was selected by setting λ = (0.05, 0.10, 0.20), ARL_0 = 370 and α_0 = 1 in the case of MAX(1,1) with parameter β = 1.0 ,θ_1 =(0.15, 0.25, 0.30) and θ_2 =(0.10, 0.25, 0.50), respectively.

Table 1 Comparison of ARL_0 between using numerical integration for MAX)2,2(process with parameter α_0 = 1 for ARL_0 = 370

λ	θ_1	θ_2	b	$H_M(u)$ (Time: seconds)	$H_G(u)$ (Time: seconds)	$H_T(u)$ (Time: seconds)	$\%Diff$
0.05	0.15	0.10	0.019481940	370.577 (4.765)	370.577 (39.703)	370.286 (5.344)	0.0785
		0.25	0.022672201	370.514 (4.781)	370.514 (38.516)	370.349 (5.344)	0.0445
		0.50	0.029210100	370.435 (4.875)	370.435 (41.36)	370.208 (5.218)	0.0613
0.10	0.25	0.10	0.043581900	370.382 (5.25)	370.382 (41.047)	370.156 (5.328)	0.0610
		0.25	0.050820500	370.627 (4.875)	370.627 (41.75)	370.402 (5.219)	0.0607
		0.50	0.065747070	370.268 (4.953)	370.268 (41.203)	370.045 (5.266)	0.0602
0.20	0.3	0.10	0.093940200	370.287 (4.875)	370.287 (41.734)	370.067 (5.500)	0.0594
		0.25	0.110015960	370.527 (4.813)	370.527 (41.265)	370.309 (5.297)	0.0588
		0.50	0.143624000	370.392 (4.796)	370.392 (41.188)	370.177 (5.430)	0.0580

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As shown in Table 1, from NIE methods using Gaussian and midpoint rules are similar values, but using trapezoidal rules yield slightly different results on m=800 subintervals. If we take into account the absolute percentage error computed by $\%Diff = \frac{H_M(u) - H_T(u)}{H_M(u)} \times 100$. It can note that ARL_0 values using the trapezoid rule $\%Diff$ are less than 0.08% compared to other methods. However, the CPU time using midpoint and trapezoidal rules are less than the CPU time using the Gaussian rule.

The ARL values indicate the performance of EWMA control chart for detection change in the process mean using Gaussian, midpoint, and trapezoidal rules for m=800 subintervals shown in Table 2 to Table 4. We set the parameter α_0=1 when process is in-control and the parameter $\alpha_1 = \alpha_0(1+\delta)$ when process is out of control where shift size (δ)=0.005, 0.02, 0.04, 0.06, 0.08, 0.1, 0.5, and 1.0. In Table 2, the initial ARL_0 = 370 for MAX(2,1) with parameters θ_1 = 0.35, θ_2 = 0.6 with= ARL_0, the initial In Table 3 =0.01761086. b= 0.05 and λ = 2.0, β 370 for MAX(2,2) with parameters θ_1 = 0.15, θ_2 = 0.25, with λ , = 0.7 β_2 5, = 0. β_1 =0.05 and b = 0.022672201. In Table 4, the initial ARL_0 = 370 for MAX(3,2) with parameters θ_1 = 0.15, θ_2 = 0.25, θ_3 = 0.45 with=0.03579842.b=0.05 and λ = 0.7, β_2 = 0.5, β_1

Table 2 ARL values for MAX(2,2) with parameters θ_1 = 0.15, θ_2 = 0.25, with= 0.7, β_2 = 0.5, β_1 = ARL_0 for =0.022672201 $b$and =0.05 and λ 370

shift (δ)	$H_M(u)$ (Time: seconds)	$H_G(u)$ (Time: seconds)	$H_T(u)$ (Time: seconds)
0	370.315 (4.172)	370.315 (37.062)	370.086 (4.703)
0.005	68.831(4.219)	68.831 (38.047)	68.789 (4.563)
0.02	20.6369 (4.125)	20.6369 (38.578)	20.6248 (4.781)
0.04	11.0660 (4.266)	11.0660 (40.781)	11.0598 (4.906)
0.06	7.75846 (4.328)	7.75846 (41.531)	7.75435 (5.031)
0.08	6.08201 (4.187)	6.08201 (41.406)	6.07893 (5.156)
0.10	5.06887 (4.187)	5.06887 (41.719)	5.06641 (5.031)
0.5	1.7953 (4.250)	1.7953 (37.531)	1.79483 (5.047)
1.0	1.38795 (4.157)	1.38795 (42.422)	1.38772 (5.125)

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Table 3 ARL values for max (2,1) using numerical integral equation method given θ_1 = 0.35, θ_2 = 0.6 with= ARL_0 for =0.01761086 b = 0.05, λ = 2.0, β 370

shift (δ)	$H_M(u)$ (Time: seconds)	$H_G(u)$ (Time: seconds)	$H_T(u)$ (Time: seconds)
0	370.5144 (4.781)	370.5144 (37.422)	370.286 (4.89)
0.005	73.8584 (4.735)	73.8584 (37.984)	73.8135 (4.906)
0.02	22.3826 (4.657)	22.3826 (38.578)	22.3695 (5.219)
0.04	12.0145 (4.735)	12.0145 (39.515)	12.0078 (4.985)
0.06	8.4196 (4.984)	8.4196 (39.172)	8.4151 (4.875)
0.08	6.59482 (4.781)	6.59482 (38.9999)	6.59145 (4.969)
0.10	5.49099 (4.625)	5.49099 (40.235)	5.48829 (5.234)
0.5	1.91019 (4.718)	1.91019 (39.391)	1.909669 (5.469)
1.0	1.45598 (4.828)	1.45598 (40.594)	1.45572 (5.343)

Table 4 ARL values for max (2,2) using numerical integral equation method given θ_1 = 0.15, θ_2 = 0.25, θ_3 = 0.45 with= ARL_0 for =0.03579842. b=0.05 and λ = 0.7, β_2 0.5, = β_1 370

Shift (δ)	$H_M(u)$ (Time: seconds)	$H_G(u)$ (Time: seconds)	$H_T(u)$ (Time: seconds)
0	370.584 (4.781)	370.584 (37.36)	370.358 (5.094)
0.005	85.2404 (4.735)	85.2404 (37.984)	85.1888 (5.172)
0.02	26.4916 (4.657)	26.4916 (38.922)	26.4761 (5.016)
0.04	14.2629 (4.735)	14.2629 (38.922)	14.2549 (5.312)
0.06	9.98991 (4.984)	9.98991 (39.860)	9.9845 (5.719)
0.08	7.8138 (4.781)	7.81380 (37.265)	7.80973 (5.515)
0.10	6.49476 (4.625)	6.49476 (37.157)	6.49149 (5.407)
0.5	2.18525 (4.718)	2.18525 (37.578)	2.18458 (5.437)
1.0	1.62088 (4.828)	1.62088 (37.375)	1.62053 (5.203)

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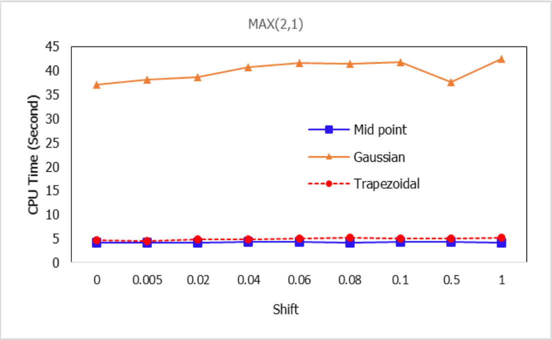
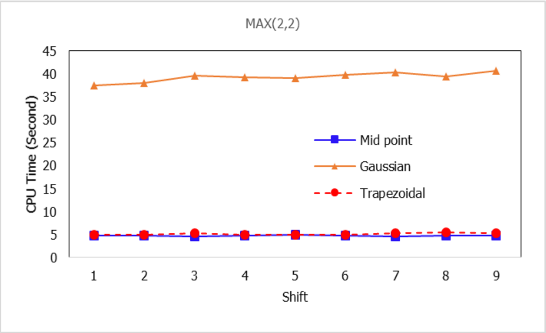
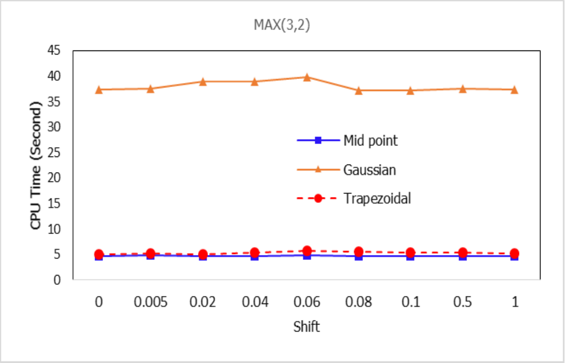


(a)



(b)



(c)

Figure 1 The CPU times for evaluating ARL values of EWMA control chart : (a) MAX(2,1) (b) MAX(2,2) (c) MAX(3,2)

Table 2 to Table 4 show that ARL_0 from the NIE method using Gaussian, midpoint and trapezoid rules on m=800 subintervals were equally effective in detecting process changes. In terms of CPU time, we discovered that the midpoint rule takes the least time to calculate, followed by the trapezoid rule and the Gaussian rule. Refer to Figures 1.

Conclusion

This paper proposes the numerical integration method using Gaussian, midpoint and trapezoid rules of ARL for moving average process with explanatory variables (MAX(q,r)) on the EWMA control chart. The results found that ARL values from the three methods are close together. The computational times for using midpoint and trapezoid rules take less than 6 seconds. Besides, using the Gaussian rule takes more CPU times, around 37 - 43 seconds. As a result, employing midpoint and trapezoid rules may significantly reduce computing time compared to the Gaussian rule.

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