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Research Article

Finite Dimensional Simple Poisson Modules of a Poisson Algebras A

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Abstract

This study focus on a Poisson algebra $A = \mathbb{C}[x, y, z]$ with a Poisson bracket $\{x, y\} = yx + z,\ \{y, z\} = zy + x,\ \{z, x\} = xz + y.$ There are precisely five Poisson maximal ideals as following.

$$J_1 = xA + yA + zA,$$
$$J_2 = (x+1)A + (y+1)A + (z+1)A,$$
$$J_3 = (x+1)A + (y-1)A + (z-1)A,$$
$$J_4 = (x-1)A + (y+1)A + (z-1)A, \text{ and}$$
$$J_5 = (x-1)A + (y-1)A + (z+1)A.$$

In this research, we determine the finite dimensional simple Poisson modules annihilated by $J_i;\ i = 1, 2, \ldots, 5$.

Keywords: Poisson algebra, Poisson module, simple Poisson module, finite-demensional simple Poisson module

Introduction

The Poisson algebra is an algebra with the brackets, called Poisson bracket, introduced in 1809 by Joseph-Louis Lagrange and his student, Siméon-Denis de Poisson, as an algorithm useful to produce solutions of motion. It is the impotant tool to study in Mathematics and Physics. There are many mathematicians study a Poisson algebra as a Poisson geometry, quantum group and deformation of commutative associative algebras. In Physics, a Poisson algebra is crucial for study the deformation quantization, Hamiltonian mechanic and topological field theories.

There are many authors who interested in a Poisson algebra. The study on a Poisson algebra as an operator algebras are wildly study. Sasom (2006) used the direct method constructed the Poisson algebra from the algebra T and the quantized enveloping algebra $U_q\left(sl_2\right)$. She found the Poisson maximal ideals of each Poisson algebra and determined the Poisson modules annihilated by each Poisson maximal ideal. After that, Jordan (2010) applied the method in Erdmann and Wildon (2006) (low-dimensional Lie algebra) to the Poisson algebra constructed from the quantized enveloping algebra $U_q\left(sl_2\right)$, a Poisson algebra B, presented in Sasom (2006). He showed that there was a unique d-dimensional simple Poisson module over B annihilated by each Poisson maximal ideals for each $d \geq 1$. Afterwards, Sasom (2012) studied the Poisson algebra $A = \mathbb{C}\left[x, y, z\right]$ with a Poisson bracket $\{x, y\} = yx + z$, $\{y, z\} = zy + x$ and $\{z, x\} = xz + y$. She found that there were precisely five Poisson maximal ideals as the following.

$$\begin{aligned}
J_1 &= xA + yA + zA, \\
J_2 &= (x+1)A + (y+1)A + (z+1)A, \\
J_3 &= (x+1)A + (y-1)A + (z-1)A, \\
J_4 &= (x-1)A + (y+1)A + (z-1)A, \\
J_5 &= (x-1)A + (y-1)A + (z+1)A.
\end{aligned}$$

She used the direct method constructed the Poisson modules which annihilated by each $J_i; i = 1, \ldots, 5$. Later, Changtong and Sasom (2018) studied the Poisson algebra $A = \mathbb{C}\left[x, y, z\right]$ with a Poisson bracket $\{x, y\} = yx + x + y + z$, $\{y, z\} = zy + x + y + z$ and $\{z, x\} = xz + x + y + z$. They found that there were only two Poisson maximal ideals $J_1 = xA + yA + zA$ and $J_2 = (x+3)A + (y+3)A + (z+3)A$. They used the same method in Jordan (2010) showed that every finite-dimensional simple Poisson module over A annihilated by J_1 was one-dimensional and for each $d \geq 1$, there was a unique d-dimensional simple Poisson module over B annihilated by J_2.

In this present paper, we focus on the Poisson algebra $A = \mathbb{C}\left[x, y, z\right]$ constructed by Sasom (2012) with the Poisson maximal ideals as above. We use the same method in Jordan (2010) and Changtong and Sasom (2018) determine the finite dimensional simple Poisson modules annihilated by each Poisson maximal ideal. This Poisson module is very important to the study in the Poisson geometry and the quantum mechanic.

Materials and methods

In this section, it contains some of the materials that will be used throughtout this research. The main topics are Lie algebras, Low-dimensional Lie algebra, Poisson algebra and Poisson module.

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Lie algebras

Definition 1. [Henderson (2012)] Let L be a vector space over a field F. A *Lie algebra* is a vector space L with a map $[-,-]: L \times L \to L$ satisfying:

(i) the map $[-,-]$ is bilinear, i.e., for all $a,b \in \mathbb{C}$ and $x,y,z \in L$

$$[ax+by,z] = a[x,z]+b[y,z],$$
$$[x,ay+bz] = a[x,y]+b[x,z];$$

(ii) the map $[-,-]$ is skew-symmetric, i.e. $[y,x] = -[x,y]$ for all $x,y \in L$;

(iii) the map $[-,-]$ satisfies the Jacobi identity, i.e.

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]] = 0 \text{ for all } x,y,z \in L.$$

So, the map $[-,-]$ is called *Lie bracket* and (ii) implies $[x,x]=0$ for all $x \in L$.

Next, we will introduce the special Lie algebras that useful in this study. Firstly, we introduce the *special linear Lie algebra* $sl(n,\mathbb{C})$ of $n \times n$ matrics with trace 0. It has dimension n^2-1. The most impotant is a Lie algebra $sl(2,\mathbb{C})$ of 2×2 matrics with trace 0. It is a three-dimensional Lie algebra with the standard basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The Lie bracket of $sl(2,\mathbb{C})$ is given by $[e,f]=h,\ [h,e]=2e,\ [h,f]=-2f$.

Theorem 2. [Henderson (2012)] For all $n \geq 2$, $sl(n,\mathbb{C})$ is simple.

For more details of $sl(2,\mathbb{C})$ see in Henderson (2012).

Another one is the *special unitary Lie algebra* $su(2,\mathbb{C})$ of 2×2 skew-Hermitean matrics with trace 0. It has the basis

$$x = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad y = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad z = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The Lie bracket of $su(2,\mathbb{C})$ is given by $[x,y]=z,\ [y,z]=x,\ [z,x]=y$. So, $su(2,\mathbb{C})$ has dimension 3 which implies that $su(2,\mathbb{C}) \cong sl(2,\mathbb{C})$.

For more details of $su(2,\mathbb{C})$ see in Pfeifer (2003) and Kirillov (2008).

Definition 3. Let L be a Lie algebra. A subalgebra H of L is *a Lie subalgebra* if $[x,y] \in H$, for all $x,y \in H$.

Definition 4. An *ideal* of a Lie algebra L is a subspace I of L such that $[x,y] \in I$ for all $x \in L,\ \ y \in I$.

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Let I, J be ideals of the Lie algebra L. The product $[I, J]$ is the span of the commutators of elements of I, J, that is $[I, J] := Span\{[x, y] | x \in I, y \in J\}$. One of the tools for investigating the results is the derived algebra. It is defined as following.

Definition 5. Let I, J be ideals of a Lie algebra L. If $I = J = L$, then the ideal $[L, L]$ is called the *derived algebra* of L.

Low-dimensional Lie algebra

The low-dimension is the basic way to find how many non-isomorphic Lie algebras in order to classify them. The reason to study on the low-dimensional Lie algebras is that there often appear as subalgebras of the larger Lie algebras. We will look at the Lie algebras of dimensions 1,2 and 3.

It is easily to see that every 1 dimensional Lie algebra is abelian. For dimension 2, we can study by using the following theorem.

Theorem 6. [Erdmann & Wildon (2006)] Let F be any field. Up to isomorphism, there is a unique two-dimensional non-abelian Lie algebra over F. This Lie algebra has a basis $\{x, y\}$ such that its Lie bracket is described by $[x, y] = x$. The centre of this Lie algebra is 0.

For 3-dimensional Lie algebras, we focus for 2 cases. Firstly, the 3 dimensional Lie algebra L with its derived algebra $[L, L]$ has dimension 1 and $[L, L] \subseteq Z(L)$, the center of L, appears uniquely and it has a basis $\{f, g, h\}$ where $[f, g] = h \in Z(L)$. This Lie algebra is known as the *Heisenberg algebras*. For another one, suppose that L is a complex Lie algebra of dimension 3 such that $[L, L] = L$. Up to isomorphism, $sl(2, \mathbb{C})$ is the unique 3-dimensional Lie algebras L with $[L, L] = L$.

For other cases and more details of low-dimensional Lie algebras, one can study in Erdmann & Wildon (2006).

Poisson algebra and Poisson module

Definition 7. A *Poisson algebra* A is a commutative algebra over a field F together with a bilinear map $\{-,-\}: A \times A \to A$ such that $(A, \{-,-\})$ is a Lie algebra and satisfies a Leibniz identity

$$\{xy, z\} = x\{y, z\} + \{x, z\}y, \text{ for all } x, y, z \in A.$$

We call $\{-,-\}$ a *Poisson bracket* on A. A subalgebra B of A is a *Poisson subalgebra* if $\{b, c\} \in B$ for all $b, c \in B$.

Definition 8. Let A be a Poisson algebra. An ideal I of A is a *Poisson ideal* if $\{i, a\} \in I$ for all $i \in I$, $a \in A$.

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Moreover, the factor A/I is also a Poisson algebra with a Poisson bracket $\{a+I, b+I\} = \{a,b\} + I$ for all $a,b \in A$.

Definition 9. Let A be a Poisson algebra. A Poisson ideal I of A, which $I \neq A$, is said to be a *Poisson maximal ideal* if it is also a maximal ideal of A.

Definition 10. Let A be a commutative Poisson algebra with a Poisson bracket $\{-,-\}$ and M be a module over A. An A-module M is called a *Poisson module* if there is a bilinear form $\{-,-\}_M : A \times M \to M$ such that

(i) $\{a, bm\}_M = \{a,b\} m + b\{a,m\}_M$,

(ii) $\{ab, m\}_M = a\{b,m\}_M + b\{a,m\}_M$,

(iii) $\{\{a,b\}, m\}_M = \{a, \{b,m\}_M\}_M + \{b, \{a,m\}_M\}_M$,

for all $a,b \in A$ and $m \in M$.

A submodule N of a Poisson module M is called a *Poisson submodule* if $\{a,n\}_M \in N$ for all $a \in A$ and $n \in N$.

Definition 11. Let M be a Poisson A-module and $J \subseteq M$. The *annihilator* of J in A, in the module sense, is denoted by $ann_A(S)$ and the set $Pann_A(J)$ is defined by:

$$Pann_A(J) = \left\{ a \in A \,\middle|\, \{a,m\}_M = 0, \forall m \in J \right\}.$$

Theorem 12. [Jordan (2010)] Let A be a finitely generated Poisson algebra and M be a Poisson A-module. Let $J = ann_A(M)$. The following statements hold.

(i) J is a Poisson ideal of A.

(ii) If M is finite-dimensional and simple then J is a maximal ideal of A.

(iii) $\square + J^2 \subseteq Pann_A(M)$.

Let $(A, \{-,-\})$ be Poisson algebra and I, J be Poisson ideals of A. Then IJ is a Poisson ideals of A. Of course I and J are Lie subalgebras of A under $\{-,-\}$. If $I \subseteq J$, then I is a Lie ideal of J and J/I is a Lie algebra. In particular, J/J^2 is always a Lie algebra. For the study on the Poisson modules, we can find that the factor J/I is a Poisson A-module with $\{a, j+I\}_{J/I} = \{a, j\}_J + I$ where I, J are Poisson ideals of A with $I \subseteq J$ and $a \in A,\ j \in J$. By the same argument, J/I is also a Lie algebra. Every Poisson subalgebra of J/I is a Lie ideal, so if J/I is simple as a Lie algebra, then it is simple as a Poisson module. If A is a finitely generated Poisson algebra and J is a Poisson maximal ideal, so that

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$A = J + \mathbb{C}$, then the converse is also true because every Lie ideal of J/I is then a Poisson A-submodule.

The next theorem is the main theorem by Jordan (2010). That is a method to determine the finite-dimensional simple Poisson modules over any affine Poisson algebra.

Theorem 13. [Jordan (2010)] Let A be a finitely generated Poisson algebra.

(i) Let M be a finite-dimensional simple Poisson A-module and $J = ann_A(M)$. There is a simple module M^* for the Lie algebra $g(J)$ such that $M^* = M$, as a vector space, and $\left[j + J^2, m\right]_{M^*} = \{j, m\}_M$ for all $j \in J$ and $m \in M$.

(ii) Let J be a Poisson maximal ideal of A and N be a finite-dimensional simple $g(J)$-module. There exist a simple Poisson A-module N' and a Lie homomorphism $f : A \to g(J)$ such that $N' = {}^f N$ as a Lie module over A and $J = ann_A(N')$.

(iii) For all finite-dimensional simple Poisson modules M, $M^{*\prime} = M$. For all Poisson maximal ideals J of A and all finite-dimensional simple $g(J)$-modules N, $N' = N$.

(iv) The procedure in (i) and (ii) establish a bijection Γ from the set if isomorphism classes of finite-dimensional simple Poisson module over A to the set of pairs $(J, \hat{N})$, where J is a Poisson maximal ideal of A, N is a finite-dimensional simple $g(J)$-module and $\hat{N}$ denote its isomorphism class, given by $\Gamma(\hat{M}) = (ann_A(M), \hat{M}^*)$.

For more details of Poisson algebra and Poisson modules see in Jordan (2010).

Results

In this section, we study on the Poisson algebra A with Poisson maximal ideals $J_i;\, i = 1, 2, \ldots, 5$. We use the method in Erdmann & Wildon (2006) to determine the finite dimensional simple Poisson modules annihilated by Poisson maximal ideals $J_i;\, i = 1, 2, \ldots, 5$. The results of the study are follows.

Lemma 14. Let A be a Poisson algebra with a Poisson maximal ideal $J_1 = xA + yA + zA$. For $d \geq 1$, there is a unique d-dimensional simple Poisson A-module annihilated by J_1.

Proof. Let A be a Poisson algebra with a Poisson maximal ideal $J_1 = xA + yA + zA$. We consider the Lie algebra $g(J_1) = J_1/J_1^2$. So, $g(J_1)$ has a basis x, y, z and the bracket $\{x, y\}$, $\{y, z\}$ and $\{z, x\}$ became $[x, y] = z,\ [y, z] = x,\ [z, x] = y$. Then $g(J_1) \cong su(2, \mathbb{R}) \cong sl(2, \mathbb{C})$. It follows that, for each $d \geq 1$, A has a unique d-dimensional simple Poisson module annihilated by J_1 .

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Lemma 15. Let A be a Poisson algebra with a Poisson maximal ideal $J_2=(x+1)A+(y+1)A+(z+1)A$. For $d\geq 1$, there is a unique d-dimensional simple Poisson A-module annihilated by J_2.

Proof. Let A be a Poisson algebra with a Poisson maximal ideal $J_2=(x+1)A+(y+1)A+(z+1)A$. We consider the Lie algebra $g(J_2)={J_2}/{J_2^2}$. Let $m=x+1,\ n=y+1,\ p=z+1$. Then $J_2=mA+nA+pA$. So, in J_2,

$$\begin{aligned}\{m,n\} &= \{x+1,y+1\}=\{x,y\}\\ &= yx+z\\ &= (n-1)(m-1)+(p-1)\\ &= nm-n-m+1+p-1\\ &= nm-n-m+p.\end{aligned}$$

Similarly, we have $\{n,p\}=pn-p-n+m$ and $\{p,m\}=mp-m-p+n$.

Thus, in $g(J_2)$, the bracket $\{m,n\},\{n,p\},\{p,m\}$ became

$$[m,n]=-n-m+p,\quad [n,p]=-p-n+m,\quad [p,m]=-m-p+n.$$

Next, we will show that $\{[m,n],[n,p],[p,m]\}$ is linearly independent. Let β_1,β_2 and β_3 be scalars such that $\beta_1[m,n]+\beta_2[n,p]+\beta_3[p,m]=0$. So,

$$\beta_1(-n-m+p)+\beta_2(-p-n+m)+\beta_3(-m-p+n)=0$$

$$(-\beta_1-\beta_2+\beta_3)n+(-\beta_1+\beta_2-\beta_3)m+(\beta_1-\beta_2-\beta_3)p=0.$$

Then we obtain that

$$\begin{aligned}-\beta_1-\beta_2+\beta_3&=0,\\ -\beta_1+\beta_2-\beta_3&=0,\\ \beta_1-\beta_2-\beta_3&=0.\end{aligned}$$

This system can be written as following.

$$\begin{pmatrix}-1 & -1 & 1\\ -1 & 1 & -1\\ 1 & -1 & -1\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Thus it has the augmented matrix $\left(\begin{array}{ccc|c}-1 & -1 & 1 & 0\\ -1 & 1 & -1 & 0\\ 1 & -1 & -1 & 0\end{array}\right)$, which has non-zero

determinant. This implies that the system has exactly one solution that is $\beta_1=\beta_2=\beta_3=0$. Hence $\{[m,n],[n,p],[p,m]\}$ is linearly independent. So, the derived algebra $\left[g\left(J_2\right), g\left(J_2\right)\right]$ has dimension 3, which implies that $g\left(J_2\right) \cong sl\left(2,\mathbb{C}\right)$. By Handerson (2012), it is a well-known that $sl\left(2,\mathbb{C}\right)$ has a unique d-dimensional simple Poisson module annihilated by J_2 for each $d \geq 1$ and $sl\left(2,\mathbb{C}\right)$ is 1-homogeneous. By Theorem 13, the result hold for A.

Lemma 16. Let A be a Poisson algebra with a Poisson maximal ideal $J_3 = \left(x+1\right)A + \left(y-1\right)A + \left(z-1\right)A$. For $d \geq 1$, there is a unique d-dimensional simple Poisson A-module annihilated by J_3.

Proof. Let A be a Poisson algebra with a Poisson maximal ideal $J_3 = \left(x+1\right)A + \left(y-1\right)A + \left(z-1\right)A$. We consider the Lie algebra $g\left(J_3\right) = J_3 / J_3^2$. Let $m = x+1,\ n = y-1,\ p = z-1$. Then $J_3 = mA + nA + pA$. So, in J_3,

$$\{m,n\} = nm - n + m + p$$

$$\{n,p\} = pn + p + n + m$$

$$\{p,m\} = mp + m - p + n.$$

Thus, in $g\left(J_3\right)$, the bracket $\{m,n\}$, $\{n,p\}$, $\{p,m\}$ became $[m,n] = -n + m + p,\ \ [n,p] = p + n + m,\ \ [p,m] = m - p + n.$

The proof is similar to Lemma 15 that $\left\{[m,n],[n,p],[p,m]\right\}$ is linearly independent. Thus, the derived algebra $\left[g\left(J_3\right), g\left(J_3\right)\right]$ has dimension 3, which implies that $g\left(J_3\right) \cong sl\left(2,\mathbb{C}\right)$. Hence A has a unique d-dimensional simple Poisson module annihilated by J_3 for each $d \geq 1$.

Lemma 17. Let A be a Poisson algebra with a Poisson maximal ideal $J_4 = \left(x-1\right)A + \left(y+1\right)A + \left(z-1\right)A$. For $d \geq 1$, there is a unique d-dimensional simple Poisson A-module annihilated by J_4.

Proof. Let A be a Poisson algebra with a Poisson maximal ideal $J_4 = \left(x-1\right)A + \left(y+1\right)A + \left(z-1\right)A$. We consider the Lie algebra $g\left(J_4\right) = J_4 / J_4^2$. Let $m = x-1,\ n = y+1,\ p = z-1$. Then $J_4 = mA + nA + pA$. So, in J_4,

$$\{m,n\} = nm + n - m + p,$$

$$\{n,p\} = pn - p + n + m, \text{ and}$$

$$\{p,m\} = mp + m + p + n.$$

Thus, in $g\left(J_4\right)$, the bracket $\{m,n\}$, $\{n,p\}$, $\{p,m\}$ became

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$[m,n]=n-m+p,\ \ [n,p]=-p+n+m,\ \ [p,m]=m+p+n.$

The proof is similar to Lemma 15 that $\{[m,n],[n,p],[p,m]\}$ is linearly independent. Thus, the derived algebra $[g(J_4),g(J_4)]$ has dimension 3, which implies that $g(J_4)\cong sl(2,\mathbb{C})$. Hence A has a unique d-dimensional simple Poisson module annihilated by J_4 for each $d\geq 1$.

Lemma 18. Let A be a Poisson algebra with a Poisson maximal ideal $J_5=(x-1)A+(y-1)A+(z+1)A$. For $d\geq 1$, there is a unique d-dimensional simple Poisson A-module annihilated by J_5.

Proof. Let A be a Poisson algebra with a Poisson maximal ideal $J_5=(x-1)A+(y-1)A+(z+1)A$. We consider the Lie algebra $g(J_5)=J_5/J_5^2$. Let $m=x-1,\ n=y-1,\ p=z+1$. Then $J_5=mA+nA+pA$. So, in J_5,

$$\{m,n\}=nm+n+m+p,$$
$$\{n,p\}=pn+p-n+m, \text{ and}$$
$$\{p,m\}=mp-m+p+n.$$

Thus, in $g(J_5)$, the bracket $\{m,n\},\{n,p\},\{p,m\}$ became

$[m,n]=n+m+p,\ \ [n,p]=p-n+m,\ \ [p,m]=p-m+n.$

The proof is similar to Lemma 15 that $\{[m,n],[n,p],[p,m]\}$ is linearly independent. Thus, the derived algebra $[g(J_5),g(J_5)]$ has dimension 3, which implies that $g(J_5)\cong sl(2,\mathbb{C})$. Hence A has a unique d-dimensional simple Poisson module annihilated by J_5 for each $d\geq 1$.

Theorem 19. Let A be a Poisson algebra with Poisson bracket

$$\{x,y\}=yx+z,\ \ \{y,z\}=zy+x,\ \ \{z,x\}=xz+y$$

and the Poisson maximal ideals $J_i;\ i=1,2,\ldots,5$. There is a unique d-dimensional simple Poisson A-module annihilated by J_i for each $d\geq 1$.

Proof. The proof of this theorem is directly obtained from Lemma 14 - Lemma 18.

Conclusion

The study on the Poisson algebra with some bracket could bring the difficulty to find its finite-dimensional simple Poisson modules. In this research, we apply the method shown in Jordan (2010) to determine the finite-dimensional simple Poisson module over a Poisson

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algebra A. We can found that there is a unique d-dimensional simple Poisson A-module annihilated by J_i for each $d \geq 1$.

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