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Research Article

Zero Inflated and Zero Truncated Negative Binomial Weighted Garima Distributions for Modeling Count Data

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Abstract

In this paper, we introduced models for zero inflated and zero truncated based on the negative binomial-weighted Garima (NB-WG) distribution. The zero inflated negative binomial-weighted Garima (ZINB-WG) distribution is a discrete probability distribution for the excessive zero counts and overdispersion which is a mixture of Bernoulli distribution and negative binomial- weighted Garima (NB-WG) distribution. Meanwhile, a new zero truncated distribution named as the zero truncated negative binomial- weighted Garima (ZTNB-WG) distribution can be used when the response variable is the set of positive integers. Some properties of the two different versions of NB-WG distribution are discussed and the estimation of the parameters is derived by maximum likelihood method (MLE). In addition, the usefulness of the proposed distributions is illustrated by real data sets.

Keywords: zero inflated distribution, zero truncated distribution, negative binomial-weighted Garima distribution, count data, maximum likelihood estimation

Introduction

Modeling of count data is used to analyze non-negative integer outcomes in different fields of study such as insurance, medical, psychology, engineering, etc. The Poisson distribution is a classical tool for count data analysis but its underlying equidispersion condition, therefore, the negative binomial (NB) distribution is a commonly used alternative to the Poisson distribution for describing count data with overdispersion. Count data may either have an excess number of zeros (zero inflation: ZI) or the situation where the value zero do not exist (zero truncation: ZT). For ZI count data, a few common examples of such data are the number of claims for each policyholder, the number of people who tested positive for HIV in each cluster, and the number of day of hospitalizations. Several probabilistic models have been proposed to model count data with excess zeros. The zero inflated Poisson (ZIP) was first introduced by Lambert (1992) for handing excess of zero counts only if there is no overdispersion. However, the excessive zeros situation can cause overdispersion, for this data the zero inflated negative binomial (ZINB), zero-inflated generalized Poisson (ZIGP), and zero inflated double Poisson (ZIDP) are more appropriate (Greene, 1994; Ridout et al., 1998; Yang et al., 2009; Phang & Loh, 2013). In addition, zero inflated mixed NB distributions have been recently defined as alternative models which often have desirable properties for modeling count data with excess

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zeros. The models contain a mixture of Bernoulli distribution and mixed NB distribution that include the zero inflated negative binomial-generalized exponential (ZINB-GE) distribution (Aryuyuen et al., 2014), zero inflated negative binomial-Crack (ZINB-CR) distribution (Saengthong et al., 2015), zero inflated negative binomial-Sushila (ZINB-S) distribution (Yamrubboon et al., 2017), and zero inflated negative binomial-Erlang (ZINB-EL) distribution (Bodhisuwan et al., 2018).

On the other hand, the ZT model, a special case of the left-truncated model with cutpoint at zero, is applied for analyzing count data except for zero. Some examples of such data include: the number of ever born children, the length of stay in the intensive care unit (ICU) once a patient is admitted, the number of European red mites on apple leaves, and the number of items in a shopper's basket at a supermarket checkout line. For modeling the data as mentioned above, typically, zero truncated Poisson (ZTP) or zero truncated negative binomial (ZTNB) is assumed. Furthermore, many studies have shown that the zero truncated mixed/compound Poisson or negative binomial are more suitable for the ZT count data than ZTP and ZTNB such as zero truncated Poisson–Lindley (ZTPL) distribution (Ghitany et al., 2008), zero truncated Poisson–Garima (ZTPG) distribution (Shanker & Shukla, 2017), zero truncated Poisson–Ishita (ZTPI) distribution (Shukla et al., 2020), zero truncated negative binomial–generalized exponential (ZTNB-GE) distribution (Bodhisuwan, 2016), and zero truncated negative binomial–Erlang (ZTNB-EL) distribution (Bodhisuwan et al., 2017).

The negative binomial-weighted Garima (NB-WG) distribution is a new mixed NB distribution that includes three-parameters introduced by Bodhisuwan & Saengthong (2020). It represents generalization of distribution for count data, including the negative binomial-Garima (NB-G), and negative binomial-size biased Garima (NB-SBG). This new distribution has a heavy tail with positive skewness and may be considered as a preferable alternative for modeling over/underdispersed count data. Therefore, in this work, we proposed the alternative zero infated and zero truncated models based on the NB-WG distribution for count data analysis.

For the rest of this paper, the constructions of zero inflated and zero truncated negative binomial – weighted Garima distributions, along with their some mathematical properties are provided in Sections 2 and 3. Sections 4 is reserved to the parameter estimation and the usefulness of the proposed distributions is illustrated in Section 5. Finally, the conclusions of the paper are presented in Section 6.

Methods

1) The zero inflated negative binomial-weighted Garima distribution

The probability mass function (pmf) of a zero inflated distribution is defined by

$$g(x)=\begin{cases}\omega+(1-\omega)\mathrm{h}(0) & ,x=0\\ (1-\omega)\mathrm{h}(x) & ,\ x=1,2,...,\end{cases} \tag{1}$$

where x is the count-valued random variable, ω is the zero inflation parameter $(0<\omega<1)$, and $h(.)$ is the pmf of the parent count model.

Now, we propose the zero inflated negative binomial-weighted Garima (ZINB-WG) distribution, which is a mixture of two distributions between Bernoulli and NB-WG distributions.

Definition 1. Let $X|\lambda$ be a random variable following a NB distribution with parameter $r>0$ and $p=\exp(-\lambda)$, i.e., $X|\lambda \sim NB(r,p=\exp(-\lambda))$. If λ is distributed as the WG distribution with positive parameters θ and β, denoted by $\lambda \sim WG(\theta,\beta)$, then X is called a random variable of the $NB-WG(r,\theta,\beta)$ distribution.

Theorem 1. If $X \sim NB-WG(r,\theta,\beta)$, then the pmf and the factorial moment of order *k* of the distribution are given as

$$h(x;r,\theta,\beta)=\frac{\theta^{\beta}}{(\theta+\beta+1)\Gamma(\beta)}\binom{r+x-1}{x}\times\sum_{j=0}^{x}\binom{x}{j}(-1)^{j}\frac{(\theta+1)(\theta+r+j)\Gamma(\beta)+\theta\Gamma(\beta+1)}{(\theta+r+j)^{\beta+1}}, \quad (2)$$

and

$$\mu_{[k]}(\mathrm{X})=\frac{\Gamma(r+k)}{\Gamma(r)}\sum_{j=0}^{k}\binom{k}{j}(-1)^{j}\frac{\theta^{\beta}\left[(\theta+1)(\theta-k+j)\Gamma(\beta)+\theta\Gamma(\beta+1)\right]}{(\theta+\beta+1)(\theta-k+j)^{\beta+1}\Gamma(\beta)}, \quad (3)$$

where x = 0,1,2,..., for r,θ and $\beta>0$.

Proof: (see Bodhisuwan & Saengthong (2020)).

Definition 2. Let X be a random variable of NB-WG distribution with pmf $h(x;r,\theta,\beta)$ defined as in (2) and $g(x)$ in (1) is a pmf of the ZI distribution. Then $g(x)$ is a pmf of ZINB-WG distribution with parameters r,θ,β and ω.

Theorem 2. If $X \sim ZINB-WG(r,\theta,\beta,\omega)$, the pmf of the distribution is given by

$$g(x;r,\theta,\beta,\omega)=\begin{cases}\omega+(1-\omega)\dfrac{\theta^{\beta}}{(\theta+\beta+1)\Gamma(\beta)}\left(\dfrac{(\theta+1)(\theta+r)\Gamma(\beta)+\theta\Gamma(\beta+1)}{(\theta+r)^{\beta+1}}\right), & x=0\\ (1-\omega)\dfrac{\theta^{\beta}}{(\theta+\beta+1)\Gamma(\beta)}\dbinom{r+x-1}{x} & \\ \quad\times\displaystyle\sum_{j=0}^{x}\binom{x}{j}(-1)^{j}\frac{(\theta+1)(\theta+r+j)\Gamma(\beta)+\theta\Gamma(\beta+1)}{(\theta+r+j)^{\beta+1}}, & x=1,2,...,\end{cases} \quad (4)$$

where shape parameters $r,\beta>0$ and scale parameters $\theta>0,\ 0<\omega<1.$

Proof: By Definition 2, the pmf of the ZINB-WG distribution is obtained by substituting (2) into (1).

Figure 1 shows some specified parameters $r,\theta,\beta,$ and ω of the ZINB-WG distribution and their graphs. From the graphs it can be observed that the distribution has a positive skew and seems to be flat when θ is decreasing or ω is increasing. In addition, the pmf of the ZINB-GE distribution can take different shapes when values of r or β differ.

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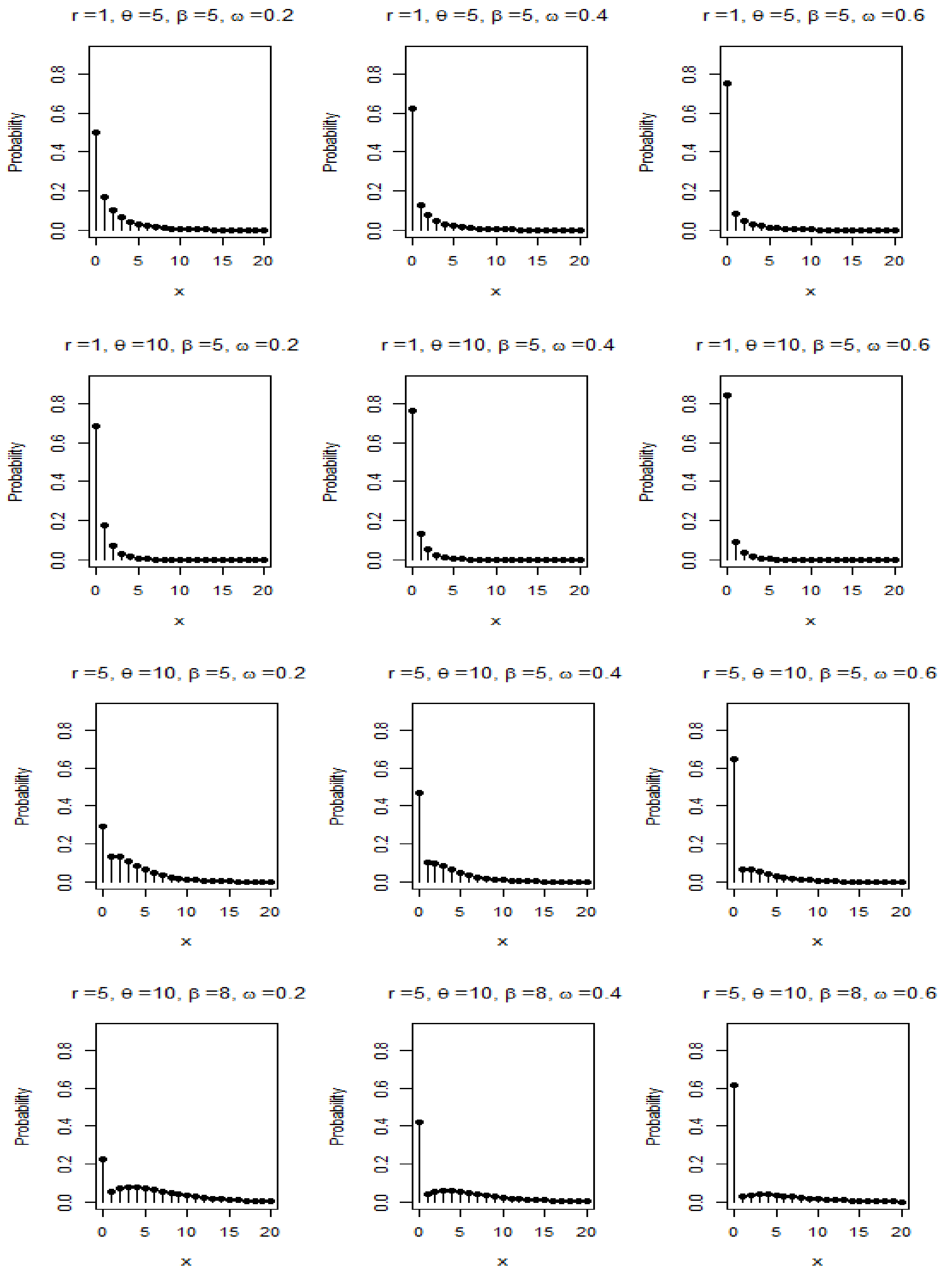

Figure 1 Some pmf plots of the ZINB-WG random variable with specified parameters r, θ, β and ω.

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Theorem 3. If $X \sim ZINB-WG(r,\theta,\beta,\omega)$, some basic properties are:

(a) The k^{th} moment of X is given by

$$E(X^k)=(1-\omega)\frac{\theta^\beta}{(\theta+\beta+1)\Gamma(\beta)}\sum_{x=1}^{\infty}x^k\binom{r+x-1}{x} \times\sum_{j=0}^{x}\binom{x}{j}(-1)^j\frac{(\theta+1)(\theta+r+j)\Gamma(\beta)+\theta\Gamma(\beta+1)}{(\theta+r+j)^{\beta+1}} \quad ,\mathrm{k}=1,2,..., \tag{5}$$

(b) The mean and variance are given as

$E(X)=(1-\omega)r\left(\frac{\theta^\beta((\theta+1)(\theta-1)+\theta\beta)}{(\theta+\beta+1)(\theta-1)^{\beta+1}}-1\right)$, and

$$Var(X)=E(X^2)-(\mathrm{E}(X))^2$$
$$=(1-\omega)\left((r^2+r)\left(\frac{\theta^\beta((\theta+1)(\theta-2)+\theta\beta)}{(\theta+\beta+1)(\theta-2)^{\beta+1}}\right)-(2r^2+r)\left(\frac{\theta^\beta((\theta+1)(\theta-1)+\theta\beta)}{(\theta+\beta+1)(\theta-1)^{\beta+1}}\right)+r^2\right)$$
$$-\left((1-\omega)r\left(\frac{\theta^\beta((\theta+1)(\theta-1)+\theta\beta)}{(\theta+\beta+1)(\theta-1)^{\beta+1}}-1\right)\right)^2.$$

2) The zero truncated negative binomial-weighted Garima distribution

The pmf of the zero truncated distribution is defined as

$$g(x)=\frac{h(x)}{1-h(0)} \qquad ,x=1,2,..., \tag{6}$$

where $h(x)$ is a pmf of the non-truncated distribution.

Definition 3. Let X be a random variable of NB-WG distribution with pmf $h(x;r,\theta,\beta)$ defined as in (2) and $g(x)$ in (6) is a pmf of the ZT distribution. Then $g(x)$ is a pmf of zero truncated negative binomial – weighted Garima (ZTNB-WG) distribution with parameters r,θ and β.

Theorem 4. If $X \sim ZTNB-WG(r,\theta,\beta)$, the pmf of the distribution is given by

$$g(x;r,\theta,\beta)=\frac{1}{1-h(0)}\left(\frac{\theta^\beta}{(\theta+\beta+1)\Gamma(\beta)}\right)\binom{r+x-1}{x} \times\sum_{j=0}^{x}\binom{x}{j}(-1)^j\frac{(\theta+1)(\theta+r+j)\Gamma(\beta)+\theta\Gamma(\beta+1)}{(\theta+r+j)^{\beta+1}}, \tag{7}$$

where x = 1,2,... , for shape parameters $r,\beta>0$, scale parameter $\theta>0$ and

$$h(0)=\frac{\theta^\beta}{(\theta+\beta+1)\Gamma(\beta)}\left(\frac{(\theta+1)(\theta+r)\Gamma(\beta)+\theta\Gamma(\beta+1)}{(\theta+r)^{\beta+1}}\right).$$

Proof: By Definition 3, substitute (2) into (6), then the pmf for the ZTNB-WG distribution can be expressed as (7).

Some characteristics and graphs of the ZTNB-WG distribution with specified values of parameters r, θ and β are provided in Figure 2. It shows that the distribution has a positively skewed and seems to be flat when θ is decreasing. Likewise, parameters r and β are those whose effect on the shape of this distribution.

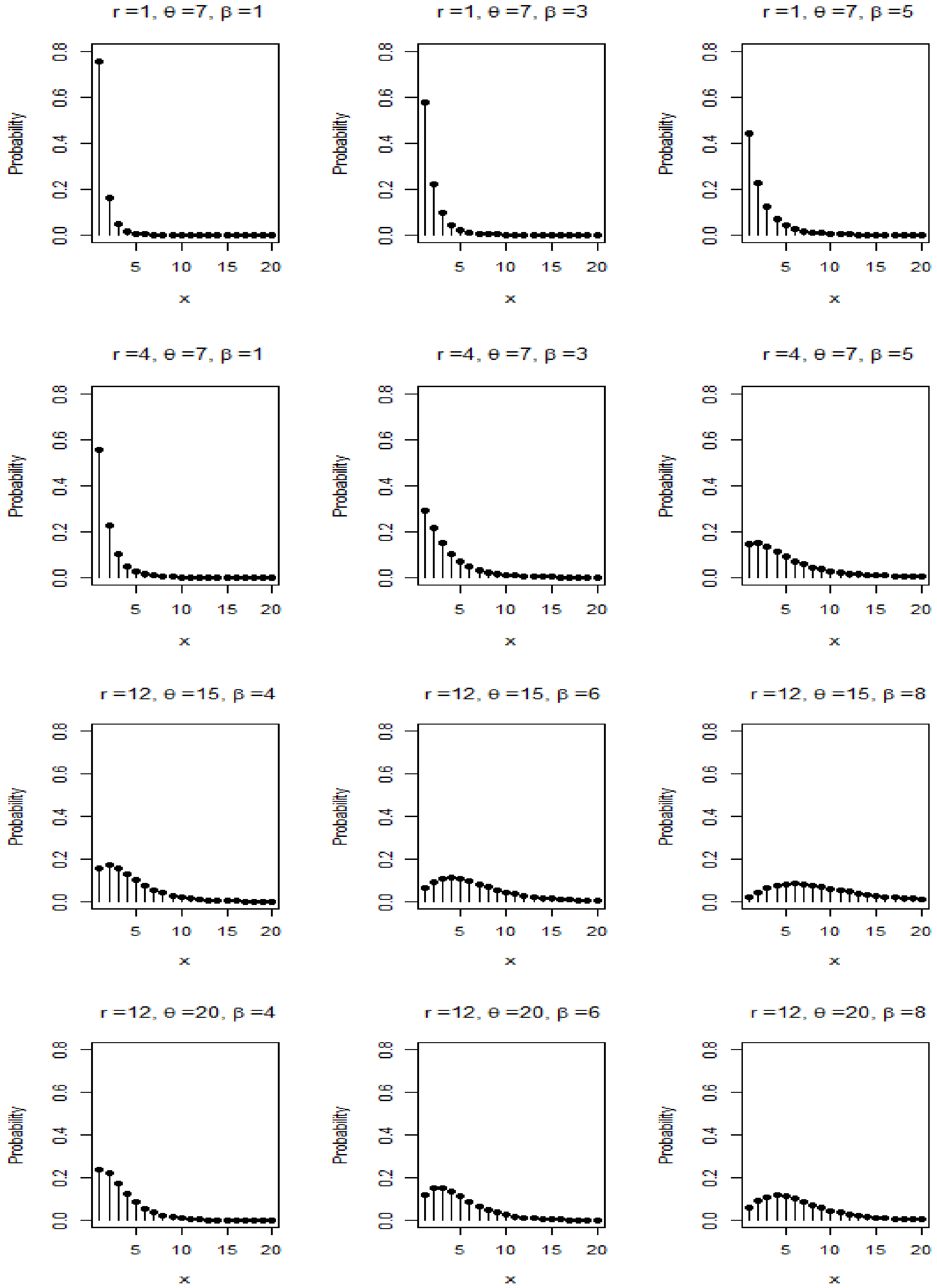

Figure 2 Some pmf plots of the ZTNB-WG random variable with specified parameters r, θ and β.

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Theorem 5. If $X \sim ZTNB-WG(r,\theta,\beta)$, some mathematical properties are:

(a) The factorial moment of order k of X is given by

$$\mu'_{[k]}(X)=\frac{1}{1-h(0)}\frac{\Gamma(r+k)}{\Gamma(r)}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{\theta^\beta\left[(\theta+1)(\theta-k+j)\Gamma(\beta)+\theta\Gamma(\beta+1)\right]}{(\theta+\beta+1)(\theta-k+j)^{\beta+1}\Gamma(\beta)}, \quad (8)$$

where k = 1,2,... , for $r,\theta,\beta>0$ and $h(0)=\frac{\theta^\beta}{(\theta+\beta+1)\Gamma(\beta)}\left(\frac{(\theta+1)(\theta+r)\Gamma(\beta)+\theta\Gamma(\beta+1)}{(\theta+r)^{\beta+1}}\right)$.

Proof: If $X|\lambda \sim NB(r,p=\exp(-\lambda))$ which is weighted by $\frac{1}{1-h(0)}$ and $\lambda \sim WG(\theta,\beta)$, then the factorial moment of order k of X can be derived from

$$\mu'_{[k]}(X)=E_\lambda\left[\frac{1}{1-h(0)}\mu_k(X|\lambda)\right]$$

$$=E_\lambda\left[\frac{1}{1-h(0)}\frac{\Gamma(r+k)}{\Gamma(r)}\frac{\left(1-e^{-\lambda}\right)^k}{e^{-\lambda k}}\right]$$

$$=\frac{1}{1-h(0)}\frac{\Gamma(r+k)}{\Gamma(r)}E_\lambda\left(e^\lambda-1\right)^k.$$

Applying the binomial expansion for the term $\left(e^\lambda-1\right)^k$, we have

$$\mu'_{[k]}(X)=\frac{1}{1-h(0)}\frac{\Gamma(r+k)}{\Gamma(r)}\sum_{j=0}^{k}\binom{k}{j}(-1)^j E_\lambda\left(e^{\lambda(k-j)}\right)$$

$$=\frac{1}{1-h(0)}\frac{\Gamma(r+k)}{\Gamma(r)}\sum_{j=0}^{k}\binom{k}{j}(-1)^j M_\lambda(k-j).$$

Based on the mgf of the WG distribution (see Bodhisuwan & Saengthong (2020), $\mu'_{[k]}(X)$ can be written as (8).

(b) The mean and variance are given by

$$E(X)=\frac{r\left[\frac{\theta^\beta\left((\theta+1)(\theta-1)+\theta\beta\right)}{(\theta+\beta+1)(\theta-1)^{\beta+1}}\right]-r}{1-h(0)},$$

and

$$Var(X)=\frac{1}{1-h(0)}\left[\left(r^2+r\right)\left(\frac{\theta^\beta\left((\theta+1)(\theta-2)+\theta\beta\right)}{(\theta+\beta+1)(\theta-2)^{\beta+1}}-\frac{2\theta^\beta\left((\theta+1)(\theta-1)+\theta\beta\right)}{(\theta+\beta+1)(\theta-1)^{\beta+1}}+1\right)\right.$$
$$\left.+r\left(\frac{\theta^\beta\left((\theta+1)(\theta-1)+\theta\beta\right)}{(\theta+\beta+1)(\theta-1)^{\beta+1}}\right)-r\right]-\frac{\left[r\left(\frac{\theta^\beta\left((\theta+1)(\theta-1)+\theta\beta\right)}{(\theta+\beta+1)(\theta-1)^{\beta+1}}\right)-r\right]^2}{\left(1-h(0)\right)^2}.$$

Parameter estimation

In this part, the estimation of parameters for $ZINB-WG(r,\theta,\beta,\omega)$ and $ZTNB-WG(r,\theta,\beta)$ via the maximum likelihood estimation (MLE) are provided.

(a) MLE for the ZINB-WG distribution

In order to find a MLE, we first need to compute the likelihood function which can be obtained by

$$L(r,\theta,\beta,\omega)=\prod_{i=1}^{n}\Bigg[I_{(x_i=0)}\left(\omega+(1-\omega)\frac{\theta^{\beta}\left((\theta+1)(\theta+r)\Gamma(\beta)+\theta\Gamma(\beta+1)\right)}{(\theta+\beta+1)(\theta+r)^{\beta+1}\Gamma(\beta)}\right)$$
$$+I_{(x_i>0)}\Bigg((1-\omega)\frac{\theta^{\beta}}{(\theta+\beta+1)\Gamma(\beta)}\binom{r+x-1}{x}$$
$$\times\sum_{j=0}^{x}\binom{x}{j}(-1)^{j}\frac{(\theta+1)(\theta+r+j)\Gamma(\beta)+\theta\Gamma(\beta+1)}{(\theta+r+j)^{\beta+1}}\Bigg)\Bigg],$$

from which we calculate the log-likelihood function

$$\log L(r,\theta,\beta,\omega)=\ell(r,\theta,\beta,\omega)$$
$$=\sum_{i=1}^{n}\Bigg[I_{(x_i=0)}\log\left(\omega+(1-\omega)\frac{\theta^{\beta}\left((\theta+1)(\theta+r)\Gamma(\beta)+\theta\Gamma(\beta+1)\right)}{(\theta+\beta+1)(\theta+r)^{\beta+1}\Gamma(\beta)}\right)$$
$$+I_{(x_i>0)}\Bigg(\log(1-\omega)+\beta\log(\theta)-\log(\theta+\beta+1)-\log\Gamma(\beta)+\log(r+x_i-1)!$$
$$-\log(r-1)!-\log(x_i)!+\log\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^{j}\frac{(\theta+1)(\theta+r+j)\Gamma(\beta)+\theta\Gamma(\beta+1)}{(\theta+r+j)^{\beta+1}}\Bigg)\Bigg].$$

It can be verified that the first partial derivatives $\ell(r,\theta,\beta,\omega)$ with respect to r,θ,β and ω we then obtain the following differential equations:

$$\frac{\partial}{\partial r}\ell(r,\theta,\beta,\omega)$$
$$=\sum_{i=1}^{n}\Bigg[I_{(x_i=0)}\left(\frac{(1-\omega)\left(\dfrac{\theta^{\beta}\left((\theta+1)w-\left((\theta+1)w+\theta\beta\right)(\beta+1)\right)}{(\theta+\beta+1)w^{\beta+2}}\right)}{\omega+(1-\omega)\dfrac{\theta^{\beta}\left((\theta+1)w\Gamma(\beta)+\theta\Gamma(\beta+1)\right)}{(\theta+\beta+1)w^{\beta+1}\Gamma(\beta)}}\right)$$
$$+I_{(x_i>0)}\Bigg(\psi(r+x_i)-\psi(r)$$

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$$+\frac{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\left\{(\theta+1)w_j^{-\beta-1}\Gamma(\beta)-(\beta+1)w_j^{-\beta-2}\left[(\theta+1)w_j\Gamma(\beta)+\theta\Gamma(\beta+1)\right]\right\}}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{(\theta+1)w_j\Gamma(\beta)+\theta\Gamma(\beta+1)}{w_j^{\beta+1}}}\Bigg)\Bigg],$$

where $\psi(k)=\frac{\Gamma'(k)}{\Gamma(k)}$ is the digamma function.

$$\frac{\partial}{\partial\theta}\ell(r,\theta,\beta,\omega)$$

$$=\sum_{i=1}^{n}\left[I_{(x_i=0)}\left(\frac{(1-\omega)\left(\left(\frac{\theta^\beta w(\beta+1)\Gamma(\beta)+\theta^\beta(\theta+1)\Gamma(\beta)+\theta^{\beta-1}w\Gamma(\beta+1)+\theta^\beta\Gamma(\beta+2)}{(\theta+\beta+1)w^{\beta+1}\Gamma(\beta)}\right)\right.}{\omega+(1-\omega)\frac{\theta^\beta\left((\theta+1)w\Gamma(\beta)+\theta\Gamma(\beta+1)\right)}{(\theta+\beta+1)w^{\beta+1}\Gamma(\beta)}}\right.\right.$$

$$\left.\frac{-\left(\frac{\left(\theta^\beta\Gamma(\beta)\left((\theta+1)w+\theta\beta\right)\right)\left(w^\beta\Gamma(\beta)\left((\theta+\beta+1)(\beta+1)+w\right)\right)}{\left((\theta+\beta+1)w^{\beta+1}\Gamma(\beta)\right)^2}\right)\Bigg)}{\omega+(1-\omega)\frac{\theta^\beta\left((\theta+1)w\Gamma(\beta)+\theta\Gamma(\beta+1)\right)}{(\theta+\beta+1)w^{\beta+1}\Gamma(\beta)}}\right)$$

$$+I_{(x_i>0)}\left(\frac{\beta}{\theta}-\frac{1}{\theta+\beta+1}+\left(\frac{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\left\{w_j^{-\beta-1}\left[w_j\Gamma(\beta)+(\theta+1)\Gamma(\beta)+\Gamma(\beta+1)\right]\right.}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{(\theta+1)w_j\Gamma(\beta)+\theta\Gamma(\beta+1)}{w_j^{\beta+1}}}\right.\right.$$

$$\left.\left.\left.-\frac{(\beta+1)w_j^{-\beta-2}\left[(\theta+1)w_j\Gamma(\beta)+\theta\Gamma(\beta+1)\right]\}}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{(\theta+1)w_j\Gamma(\beta)+\theta\Gamma(\beta+1)}{w_j^{\beta+1}}}\right)\right)\right],$$

$$\frac{\partial}{\partial\beta}\ell(r,\theta,\beta,\omega)$$

$$=\sum_{i=1}^{n}\left[I_{(x_i=0)}\left(\frac{(1-\omega)\left(\left(\frac{\theta^\beta\left((\theta+1)w\left(\psi(\beta)+\log(\theta)\right)+\theta\beta\left(\psi(\beta+1)+\log(\theta)\right)\right)}{(\theta+\beta+1)w^{\beta+1}}\right)\right.}{\omega+(1-\omega)\frac{\theta^\beta\left((\theta+1)w\Gamma(\beta)+\theta\Gamma(\beta+1)\right)}{(\theta+\beta+1)w^{\beta+1}\Gamma(\beta)}}\right.\right.$$

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$$-\left.\left.\frac{\left(\frac{\theta^{\beta}\left((\theta+1)w\Gamma(\beta)+\theta\Gamma(\beta+1)\right)\left((\theta+\beta+1)\psi(\beta)+(\theta+\beta+1)\log w+1\right)}{(\theta+\beta+1)^{2}w^{\beta+1}\Gamma(\beta)}\right)}{\omega+(1-\omega)\frac{\theta^{\beta}\left((\theta+1)w\Gamma(\beta)+\theta\Gamma(\beta+1)\right)}{(\theta+\beta+1)w^{\beta+1}\Gamma(\beta)}}\right)\right)$$

$$+I_{(x_i>0)}\left(\log(\theta)-\frac{1}{\theta+\beta+1}-\psi(\beta)\right.$$

$$+\left(\frac{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\left\{w_j^{-\beta-1}\left[(\theta+1)w_j\Gamma(\beta)\psi(\beta)+\theta\Gamma(\beta+1)\psi(\beta+1)\right]\right.}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{(\theta+1)w_j\Gamma(\beta)+\theta\Gamma(\beta+1)}{w_j^{\beta+1}}}\right.$$

$$\left.\left.-\frac{w_j^{-\beta-1}\left[(\theta+1)w_j\Gamma(\beta)+\theta\Gamma(\beta+1)\right]\log w_j\right\}}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{(\theta+1)w_j\Gamma(\beta)+\theta\Gamma(\beta+1)}{w_j^{\beta+1}}}\right)\right],$$

and

$$\frac{\partial}{\partial\omega}\ell(r,\theta,\beta,\omega)$$

$$=\sum_{i=1}^{n}\left[I_{(x_i=0)}\left(\frac{1-\left(\frac{\theta^{\beta}\left((\theta+1)(\theta+r)\Gamma(\beta)+\theta\Gamma(\beta+1)\right)}{(\theta+\beta+1)(\theta+r)^{\beta+1}\Gamma(\beta)}\right)}{\omega+(1-\omega)\frac{\theta^{\beta}\left((\theta+1)(\theta+r)\Gamma(\beta)+\theta\Gamma(\beta+1)\right)}{(\theta+\beta+1)(\theta+r)^{\beta+1}\Gamma(\beta)}}\right)-I_{(x_i>0)}\left(\frac{1}{1-\omega}\right)\right],$$

where $w=r+\theta$ and $w_j=r+\theta+j$.

These differential equations cannot be solved analytically, as they need to rely on Newton-Raphson method to approximate MLE. The ML estimates of $\hat{r},\hat{\theta},\hat{\beta}$ and $\hat{\omega}$ can be obtained by using numerical optimization with the nonlinear minimization (nlm) function in R package, namely stats (R Core Team, 2020). Therefore, we use the negative log-likelihood function as minimizing that is equivalent to maximizing the log-likelihood function.

(b) MLE for the ZTNB-WG distribution

The likelihood function of the $ZTNB-WG(r,\theta,\beta)$ distribution is given by

$$L(r,\theta,\beta)=\frac{\theta^{\beta}}{(1-h(0))(\theta+\beta+1)\Gamma(\beta)}\prod_{i=1}^{n}\binom{r+x_i-1}{x_i}$$

$$\times\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{(\theta+1)(\theta+r+j)\Gamma(\beta)+\theta\Gamma(\beta+1)}{(\theta+r+j)^{\beta+1}},$$

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where x = 1,2,... , for $r,\theta,\beta>0$ and $h(0)=\frac{\theta^{\beta}}{(\theta+\beta+1)\Gamma(\beta)}\left(\frac{(\theta+1)(\theta+r)\Gamma(\beta)+\theta\Gamma(\beta+1)}{(\theta+r)^{\beta+1}}\right).$

The log-likelihood function can be calculated as

$$
\begin{aligned}
&\log L(r,\theta,\beta)=\ell(r,\theta,\beta)\\
&=\sum_{i=1}^{n}\left[\log(r+x_i-1)!-\log x_i!-\log(r-1)!\right]+n\beta\log(\theta)-n\log(\theta+\beta+1)\\
&\quad -n\log\Gamma(\beta)+\sum_{i=1}^{n}\left[\log\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{(\theta+1)(\theta+r+j)\Gamma(\beta)+\theta\Gamma(\beta+1)}{(\theta+r+j)^{\beta+1}}\right]\\
&\quad -n\log\left(1-\frac{\theta^{\beta}(\theta+1)(\theta+r)\Gamma(\beta)+\theta\Gamma(\beta+1)}{(\theta+\beta+1)(\theta+r)^{\beta+1}\Gamma(\beta)}\right).
\end{aligned}
$$

The first partial derivative of $\ell(r,\theta,\beta)$ with respect to r,θ and β are given as follows:

$$
\begin{aligned}
&\frac{\partial}{\partial r}\ell(r,\theta,\beta)\\
&=\sum_{i=1}^{n}\psi(r+x_i)-n\psi(r)\\
&\quad+\sum_{i=1}^{n}\left(\frac{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\left\{(\theta+1)w_j^{-\beta-1}\Gamma(\beta)-(\beta+1)w_j^{-\beta-2}\left[(\theta+1)w_j\Gamma(\beta)+\theta\Gamma(\beta+1)\right]\right\}}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{(\theta+1)w_j\Gamma(\beta)+\theta\Gamma(\beta+1)}{w_j^{\beta+1}}}\right)\\
&\quad+n\left(\frac{\frac{\theta^{\beta}\left((\theta+1)w-\left((\theta+1)w+\theta\beta\right)(\beta+1)\right)}{(\theta+\beta+1)w^{\beta+2}}}{1-\frac{\theta^{\beta}(\theta+1)(\theta+r)\Gamma(\beta)+\theta\Gamma(\beta+1)}{(\theta+\beta+1)(\theta+r)^{\beta+1}\Gamma(\beta)}}\right),
\end{aligned}
$$

where $\psi(k)=\frac{\Gamma'(k)}{\Gamma(k)}$ is the digamma function.

$$
\begin{aligned}
&\frac{\partial}{\partial\theta}\ell(r,\theta,\beta)\\
&=\frac{\mathrm{n}\beta}{\theta}-\frac{n}{\theta+\beta+1}\\
&\quad+\sum_{i=1}^{n}\left(\frac{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\left\{w_j^{-\beta-1}\left[w_j\Gamma(\beta)+(\theta+1)\Gamma(\beta)+\Gamma(\beta+1)\right]\right.}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{(\theta+1)w_j\Gamma(\beta)+\theta\Gamma(\beta+1)}{w_j^{\beta+1}}}\right.
\end{aligned}
$$

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$$-\frac{(\beta+1)w_j^{-\beta-2}\left[(\theta+1)w_j\Gamma(\beta)+\theta\Gamma(\beta+1)\right]\}}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{(\theta+1)w_j\Gamma(\beta)+\theta\Gamma(\beta+1)}{w_j^{\beta+1}}}\Bigg)$$

$$+n\left(\frac{\left(\frac{\theta^\beta w(\beta+1)\Gamma(\beta)+\theta^\beta(\theta+1)\Gamma(\beta)+\theta^{\beta-1}w\Gamma(\beta+1)+\theta^\beta\Gamma(\beta+2)}{(\theta+\beta+1)w^{\beta+1}\Gamma(\beta)}\right)}{1-\frac{\theta^\beta(\theta+1)(\theta+r)\Gamma(\beta)+\theta\Gamma(\beta+1)}{(\theta+\beta+1)(\theta+r)^{\beta+1}\Gamma(\beta)}}\right.$$

$$\left.-\frac{\left(\frac{\left(\theta^\beta\Gamma(\beta)\left((\theta+1)w+\theta\beta\right)\right)\left(w^\beta\Gamma(\beta)\left((\theta+\beta+1)(\beta+1)+w\right)\right)}{\left((\theta+\beta+1)w^{\beta+1}\Gamma(\beta)\right)^2}\right)}{1-\frac{\theta^\beta(\theta+1)(\theta+r)\Gamma(\beta)+\theta\Gamma(\beta+1)}{(\theta+\beta+1)(\theta+r)^{\beta+1}\Gamma(\beta)}}\right),$$

and

$$\frac{\partial}{\partial\beta}\ell(r,\theta,\beta)$$

$$=\mathrm{n}\log(\theta)-\frac{n}{\theta+\beta+1}-n\psi(\beta)$$

$$+\sum_{i=1}^{n}\left(\frac{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\left\{w_j^{-\beta-1}\left[(\theta+1)w_j\Gamma(\beta)\psi(\beta)+\theta\Gamma(\beta+1)\psi(\beta+1)\right]\right.}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{(\theta+1)w_j\Gamma(\beta)+\theta\Gamma(\beta+1)}{w_j^{\beta+1}}}\right.$$

$$\left.-\frac{w_j^{-z-1}\left[(\theta+1)w_j\Gamma(\beta)+\theta\Gamma(\beta+1)\right]\log w_j\}}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{(\theta+1)w_j\Gamma(\beta)+\theta\Gamma(\beta+1)}{w_j^{\beta+1}}}\right)$$

$$+n\left(\frac{\left(\frac{\theta^\beta\left((\theta+1)w\left(\psi(\beta)+\log(\theta)\right)+\theta\beta\left(\psi(\beta+1)+\log(\theta)\right)\right)}{(\theta+\beta+1)w^{\beta+1}}\right)}{1-\frac{\theta^\beta(\theta+1)(\theta+r)\Gamma(\beta)+\theta\Gamma(\beta+1)}{(\theta+\beta+1)(\theta+r)^{\beta+1}\Gamma(\beta)}}\right.$$

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$$-\frac{\left(\frac{\theta^{\beta}\left((\theta+1)w\Gamma(\beta)+\theta\Gamma(\beta+1)\right)\left((\theta+\beta+1)\psi(\beta)+(\theta+\beta+1)\log w+1\right)}{(\theta+\beta+1)^{2}w^{\beta+1}\Gamma(\beta)}\right)}{1-\frac{\theta^{\beta}(\theta+1)(\theta+r)\Gamma(\beta)+\theta\Gamma(\beta+1)}{(\theta+\beta+1)(\theta+r)^{\beta+1}\Gamma(\beta)}}\Bigg),$$

where $w=r+\theta$ and $w_j=r+\theta+j$.

The ML estimators of the ZTNB-WG distribution are obtained by solving the above partial derivative equations by using the nlm function in R package.

Results and discussion

In this section, we discuss applications of the ZINB-WG and ZTNB-WG distributions to four real data sets. The first and second data sets are considered to fit with the ZIP, ZINB, and ZINB-WG distributions, while the third and fourht data sets are used to illustrate the flexibility of the ZTP, ZTNB, and ZTNB-WG distributions.

The first data set is studied on the number of major derogatory reports (MDRs) in the credit history of individual credit card applicants (see Greene (1994)). As shown in Table 1, the percentage of zero in the number of MDRs is 80.36 and the variance-to-mean ratio or index of dispersion (ID) is 3.298, indicating that there is overdispersion (ID > 1) with a mean = 0.456, variance = 1.810 and a high percentage of zeros. Additionally, the second data set is due to Klugman et al. (2019) regarding the number of runs scored in each half-inning of World Series baseball games played from 1947 through 1960, which is displayed in Table 2. This data has 72.97% of zeros, mean = 0.484, and variance = 1.075. Likewise, it has overdispersion with an index of dispersion equal to 2.220.

The third data set contains of the number of households having at least one migrant from households according to the number of migrants (see Singh & Yadava (1981)). Based on mean = 1.547, variance = 0.819 and ID = 0.529, it can be indicated that this dataset presents an underdispersed data (ID < 1). The fitting results are shown in Table 3. Lastly, we consider the fourth data set in Table 4 which reports the number of mothers of the rural area with at least two live births by the number of infant and child deaths (see Shanker (2016)), in which the mean, variance, and ID are 1.403, 0.598, and 0.426 respectively (underdispersion).

According to the results of the distribution fitting, the best distribution is the distribution that corresponds to lower values of negative log-likelihood and a higher p-value of chi-square goodness of fit test. From the results in Table 1-4, we can conclude that the ZINB-WG and ZTNB-WG distributions are competing well against the others and gives a satisfactory fit.

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Table 1 observed and expected frequencies for the consumer credit behavior data.

No. of MDRs	Observed	Fitting distributions		
		ZIP	ZINB	ZINB-WG
0	1060	1059.99	1060.08	1060.08
1	137	80.32	94.17	132.96
2	50	80.90	72.86	54.34
3	24	54.31	45.17	27.41
4	17	27.35	24.52	15.38
5	11	11.02	12.18	9.25
6	5	5.11 (6, 7, 8-14 combined)	10.02 (6, 7, 8-14 combined)	5.85
7	6			13.72 (7, 8-14 combined)
8-14	9			
Estimated parameters		$\hat{\lambda}=2.0142$ $\hat{\omega}=0.7734$	$\hat{r}=3.9566$ $\hat{p}=0.6878$ $\hat{\omega}=0.7459$	$\hat{r}=8.0652$ $\hat{\theta}=5.0944$ $\hat{\beta}=0.2664$ $\hat{\omega}=0.1932$
Negative Log-Likelihood		1140.513	1091.034	1055.526
Chi-squares		116.076	48.936	1.635
Degree of freedom		4	3	3
p-value		<0.0001	<0.0001	0.6514

Table 2 observed and expected frequencies for the runs scored data.

No. of Runs	Observed	Fitting distributions		
		ZIP	ZINB	ZINB-WG
0	1023	793.88	1023.01	1022.69
1	222	401.81	196.05	221.71
2	87	156.41	107.72	86.24
3	32	40.59	47.41	37.15
4	18	9.31 (4, 5, 6, 7+ combined)	18.28	17.07
5	11		9.54 (5, 6, 7+ combined)	8.23
6	6			8.91 (6, 7+ combined)
7+	3			

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Estimated parameters	$\hat{\lambda}=1.3064$ $\hat{\omega}=0.6293$	$\hat{r}=3.9622$ $\hat{p}=0.7785$ $\hat{\omega}=0.5703$	$\hat{r}=12.4731$ $\hat{\theta}=15.3552$ $\hat{\beta}=0.6626$ $\hat{\omega}=0.1981$
Negative Log-Likelihood	1311.282	1293.623	1283.016
Chi-squares	267.630	23.915	1.705
Degree of freedom	2	2	2
p-value	0	<0.0001	0.4264

Table 3 observed and expected frequencies for the migration data.

No. of Migrant	Observed	Fitting distributions		
		ZTP	ZTNB	ZTNB-WG
1	375	353.98	363.88	376.19
2	143	167.70	155.13	141.38
3	49	52.96	51.46	48.24
4	17	15.36 (4–8)	19.53 (4–8)	16.04
5	2			8.15 (5–8)
6	2			
7	1			
8	1			
Estimated parameters		$\hat{\lambda}=0.9475$	$\hat{r}=4.9786$ $\hat{p}=0.8574$	$\hat{r}=4.5082$ $\hat{\theta}=32.8190$ $\hat{\beta}=4.4986$
Negative Log-Likelihood		601.645	595.277	592.732
Chi-squares		8.987	2.022	0.659
Degree of freedom		2	1	1
p-value		0.0112	0.1534	0.4168

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Table 4 observed and expected frequencies for the mortality data.

No. of Infant and Child Deaths	Observed	Fitting distributions		
		ZTP	ZTNB	ZTNB-WG
1	745	708.87	722.10	746.64
2	212	255.11	236.22	204.63
3	50	61.21	61.83	59.87
4	21	12.81 (4–6)	17.85 (4–6)	18.27
5	7			8.58 (5–6)
6	3			
Estimated parameters		$\hat{\lambda}=0.7198$	$\hat{r}=3.9940$, $\hat{p}=0.8690$	$\hat{r}=0.7688$, $\hat{\theta}=78.2835$, $\hat{\beta}=28.4131$
Negative Log-Likelihood		889.844	878.842	872.491
Chi-squares		37.003	15.169	2.537
Degree of freedom		2	1	1
p-value		<0.0001	<0.0001	0.1112

Conclusion

This paper presents zero modified versions of the NB-WG distribution, which are ZINB-WG and ZTNB-WG distributions. Some mathematical properties involving mean, variance, and k^{th} factorial moment of the proposed distributions have been derived. The parameter estimation via the maximum likelihood method is also implemented. Finally, the efficiency of the ZINB-WG and ZTNB-WG distributions compared to other traditional probability distributions are illustrated by real data sets.

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