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Research Article

Parameter estimation methods for the Marshall-Olkin Power Lomax distribution and its applications

Areeya Sudsuk [1] and Darika Yamrubboon [1*]

[1]Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok, 65000, Thailand
***E-mail**: darikay@nu.ac.th
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Abstract

The Marshall-Olkin Power Lomax (MOPLx) distribution, being extended the Power Lomax distribution, is obtained by using Marshall-Olkin (MO) scheme. This distribution can be applied for skewed data set and used as an alternative to Gamma or Weibull distributions. The main objective of this paper is to compare common method, maximum likelihood estimator (MLE) with the minimum Anderson-Darling estimator (MADE), for the shape and scale parameters of the MOPLx distribution. The efficiency of both estimators for MOPLx distribution is examined via a simulation study with different sample sizes and its specified parameters by using bias, standard deviation and root mean square error as the criteria for the comparison between MLE and MADE. Furthermore, for application of the MOPLx distribution, four life time data sets are exemplified to evaluate the performances of these estimators in both small and large samples.

Keywords: maximum likelihood estimator, minimum Anderson-Darling estimator, the Marshall-Olkin Power Lomax, simulation study

Introduction

Lifetime distributions play a central role in parametric inferences for modeling the univariate data in order to explain or predict a lifetime phenomenon. It involves the following two essential steps, determining the best fitting distribution and selecting its parameter estimation, because different distributions or parameter estimations have an effect on the result of the data analysis.

In terms of distributions, the fact that each distribution has different characteristics, so most well-known traditional distributions may not fit well on some data. As a result, the new distributions that are extended forms of traditional distributions have been proposed using several techniques in the last few years. Moreover, it was found that they are more flexible to model various real world data in a wide variety of areas such as lifetime analysis, economics, engineering, finance, insurance, biological and medical science (Akhter et al., 2020; Khaleel et al., 2020).

Recently, the Marshall-Olkin Power Lomax (MOPLx) distribution with four parameters; three shape and one scale parameters was proposed by Haq et al. (2020a). The MOPLx distribution is extended form of the Power Lomax distribution based on the Marshall-Olkin (MO) family by adding a new parameter, called tilt parameter (Cordeiro, 2014). In their work, the proposed distribution was focused on the shape characteristic of the probability density function

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(pdf), as well as hazards function, explicit forms of moments, order statistics, Rényi entropy, probability weighted moments, a parameter estimation, and its application. Additionally, the MOPLx distribution can be considered in skewed data set, and it has been used to model the wind speed (Haq et al., 2020b), lifetime and reliability (Haq et al., 2020a) data sets.

In case of parameter estimation, there are lots of methods to estimate the unknown parameters such as method of moments, least squares method, probability plotting, the expectation-maximization algorithm, Bayes estimator, or the maximum likelihood estimator (MLE). Nevertheless, the MLE is the most widely used as baseline method in the framework of various proposed distributions, including the MOPLx distribution. In addition, some newly works on distributions using MLE are illustrated: the Marshall-Olkin Gambel-Lomax distribution (Nwezza & Ugwuowo, 2020), the Marshall-Olkin Exponential Gompertz distribution (Khaleel et al., 2020) and Type II Topp Leone Power Lomax Distribution (Aryuyuen & Bodhisuwan, 2020).

In this paper, we would like to apply the minimum Anderson-Darling estimator (MADE) in MOPLx distribution and compare with MLE. The MADE has a high performance in the case of many location scale distributions and offers a better solution than MLE in special estimation problems which can occur in modelling complex phenomena (Raschke, 2017). Additionally, the MADE was used in some recent researches by Dey et al. (2017; 2019) and ZeinEldin (2019a; 2019b)

The rest of this paper is organized as following, firstly, background of the MOPLx distribution including its pdf, the cumulative distribution function (cdf), some essential mathematical properties and parameter estimation using MLE are studied. Secondly, MADE for the MOPLx distribution is discussed and its Anderson-Darling distance function is also derived. Thirdly, the efficiency of its estimators are compared by simulation study. After that, MLE and MADE are applied to the real data sets. Finally, main results of this study are summarized.

Materials and Methods

1. The MOPLx Distribution

The MOPLx distribution with three scale parameters α, λ, γ and the shape parameter β was introduced by Haq et al. (2020a). The pdf, cdf and some plots of this distribution are described as follows.

*Definition 1***.** Let X be a random variable of the MOPLx distribution with parameter $\alpha, \beta, \lambda,$ and γ. It wll be denoted as $X \sim \mathrm{MOPLx}(\alpha,\beta,\lambda,\gamma)$.

*Theorem 1***.** Let $X \sim \mathrm{MOPLx}(\alpha,\beta,\lambda,\gamma)$, then its pdf is

$$f(x) = \frac{\gamma\alpha\beta\lambda^{\alpha} x^{\beta-1}\left(\lambda + x^{\beta}\right)^{-\alpha-1}}{\left(1-(1-\gamma)\left(\lambda^{\alpha}\left(\lambda + x^{\beta}\right)^{-\alpha}\right)\right)^{2}}, \quad x \geq 0, \tag{1}$$

and the corresponding cdf is

$$F(x) = \frac{1-\lambda^{\alpha}\left(\lambda + x^{\beta}\right)^{-\alpha}}{1-(1-\gamma)\left(\lambda^{\alpha}\left(\lambda + x^{\beta}\right)^{-\alpha}\right)}, \quad x \geq 0, \tag{2}$$

where $\alpha > 0,\ \beta > 0,\ \lambda > 0,$ and $\gamma > 0.$

Some plots of the MOPLx pdf and associated cdf with various values of its parameters are shown in Figure 1. It can be seen that the pdf can be categorized into two characteristics according to the shape parameter, β;

1) When $\beta \le 1$, the pdf is decreasing.

2) When $\beta > 1$, the pdf is unimodal with right–skewed.

For the MOPLx pdf with unimodal, if the parameter α is increasing and others are fixed, the shapes get more extremely right skewed, as shown in Figure 1(c). On the other hand, the shape parameter becomes more and more flat with right skewed when either λ or γ is increasing, as displayed in Figure 1(e).

2. The mathematical properties

In this section, some essential mathematical properties of the MOPLx distribution for this study including quantile function and the rth moment about origin (Haq et al., 2020a) are presented.

The quantile function

The quantile function, the reverse of its cdf, is beneficial towards both statistical inference and application in probability. In this section, the MOPLx quantile function is shown.

Theorem 2. The quantile function of the MOPLx distribution is

$$Q(u;\alpha,\beta,\lambda,\gamma)=\left(\lambda\left(\left(\frac{1-u}{1-(1-\gamma)u}\right)^{\frac{-1}{\alpha}}-1\right)\right)^{\frac{1}{\beta}}, \tag{3}$$

where u is random variable from uniform distribution bounded on (0,1). Moreover, the median of X is given by $Q(1/2;\alpha,\beta,\lambda,\gamma)$.

The r*th moments*

The various moments are a vital class of expectation describing the important shapes or features of any distribution (e.g., mean, variance , skewness and kurtosis).

Theorem 3. Suppose $X \sim \mathrm{MOPLx}(\alpha,\beta,\lambda,\gamma)$ then the rth moment of X is given by

$$\mu_r^{'}=\alpha\gamma\sum_{k=0}^{\infty}(k+1)(1-\gamma)^k\lambda^{\frac{r}{\beta}}B\left[\frac{r}{\beta}+1,\left(\alpha(\mathrm{k}+1)-\frac{r}{\beta}\right)\right], \tag{4}$$

where $B(.,.)$ is beta function.

3. Methods of parameter estimations for the MOPLx distribution

The parameter estimation is used to estimate unknown parameters so as to describe behavior of data. In this part, the fundamental aspect of two methods of parameter estimation are described as follows.

The maximum likelihood estimator

The maximum likelihood estimation (MLE) has been known as one of the most well-known method to estimate parameters of a distribution by maximising the log-likelihood function of parameters, ℓ.

Theorem 4. Let $X_1, X_2, ..., X_n$ be an independent and identically distributed (iid) random variable corresponding to $x_1, \mathrm{x}_2, ..., x_n$ which is a random sample of size n from a population with pdf in eqution (1) and let $\mathbf{\Theta}=(\alpha,\beta,\lambda,\gamma)^T$ be the vectors of the parameters. Then, its log-likelihood function (Haq et al., 2020a) can be expressed as

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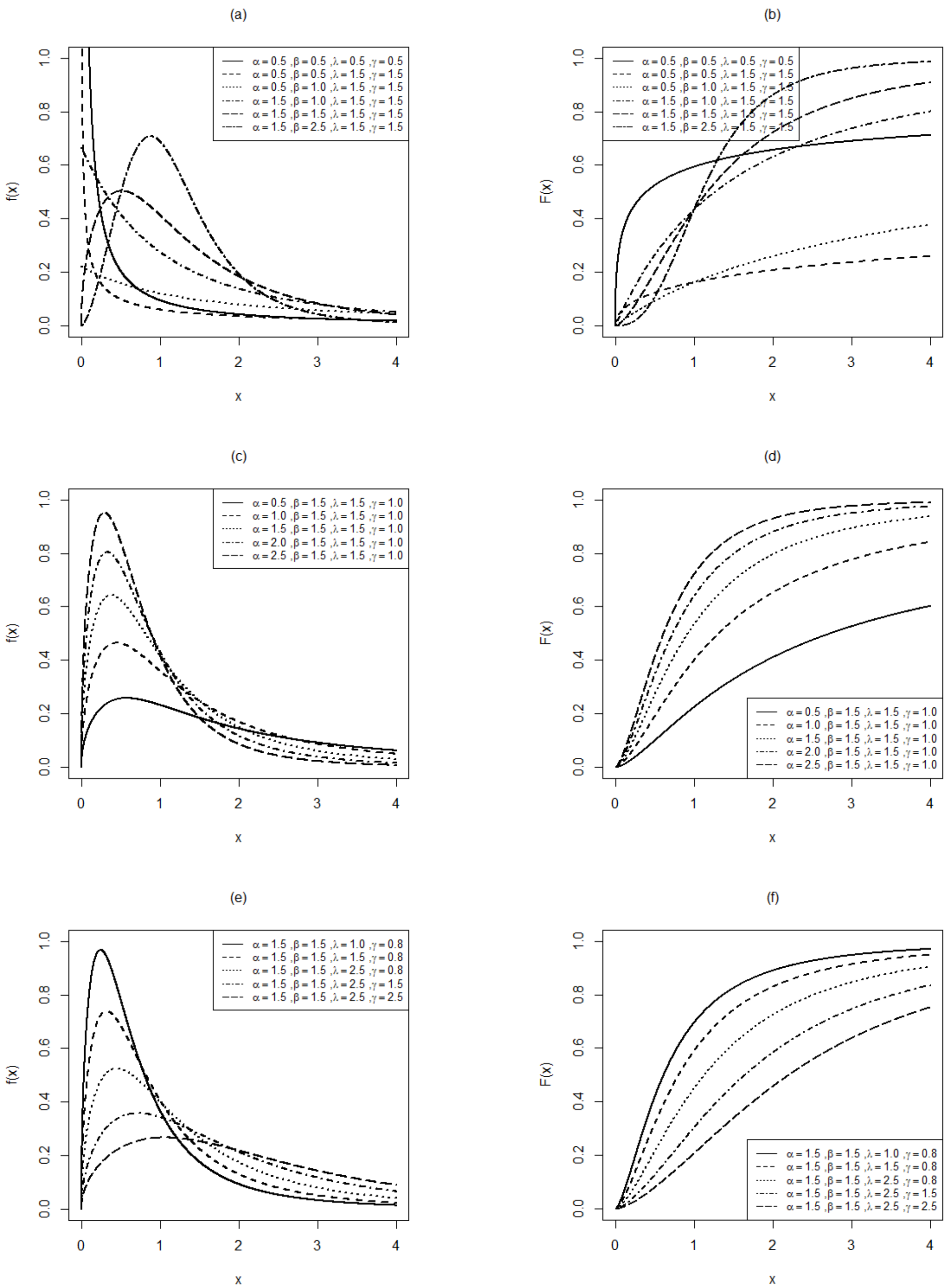

Figure 1 Some MOPLx pdf and cdf plots with its specified parameters: α, β, λ, and γ.

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$$\ell(\mathbf{\Theta}) = n\log\left(\gamma\alpha\beta\lambda^{\alpha}\right) + (\beta-1)\sum_{i=1}^{n}\log[x_i] - (\alpha+1)\sum_{i=1}^{n}\log\left[\lambda + x_i^{\beta}\right]$$
$$-2\sum_{i=1}^{n}\log\left[1-(1-\gamma)\lambda^{\alpha}\left[\lambda+x_i^{\beta}\right]^{-\alpha}\right]. \quad (5)$$

From equation (5), the associated score functions are

$$\frac{\partial\ell(\mathbf{\Theta})}{\partial\alpha} = \frac{n}{\alpha} + n\log(\lambda) - \sum_{i=1}^{n}\log\left[\lambda + x_i^{\beta}\right]$$
$$-2\sum_{i=1}^{n}\frac{-(1-\gamma)\lambda^{\alpha}\log[\lambda]\left[\lambda+x_i^{\beta}\right]^{-\alpha} + (1-\gamma)\lambda^{\alpha}\log\left[\lambda+x_i^{\beta}\right]\left[\lambda+x_i^{\beta}\right]^{-\alpha}}{1-(1-\gamma)\lambda^{\alpha}\left[\lambda+x_i^{\beta}\right]^{-\alpha}},$$

$$\frac{\partial\ell(\mathbf{\Theta})}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log[x_i] - (\alpha+1)\sum_{i=1}^{n}\frac{\log\left[x_i\right]x_i^{\beta}}{\lambda+x_i^{\beta}} - 2\sum_{i=1}^{n}\frac{\alpha(1-\gamma)\lambda^{\alpha}\log\left[x_i\right]x_i^{\beta}\left[\lambda+x_i^{\beta}\right]^{-1-\alpha}}{1-(1-\gamma)\lambda^{\alpha}\left[\lambda+x_i^{\beta}\right]^{-\alpha}},$$

$$\frac{\partial\ell(\mathbf{\Theta})}{\partial\lambda} = \frac{n\alpha}{\lambda} - (\alpha+1)\sum_{i=1}^{n}\frac{1}{\lambda+x_i^{\beta}} - 2\sum_{i=1}^{n}\frac{\alpha(1-\gamma)\lambda^{\alpha}\left[\lambda+x_i^{\beta}\right]^{-1-\alpha} - \alpha(1-\gamma)\lambda^{-1+\alpha}\left[\lambda+x_i^{\beta}\right]^{-\alpha}}{1-(1-\gamma)\lambda^{\alpha}\left[\lambda+x_i^{\beta}\right]^{-\alpha}},$$

$$\frac{\partial\ell(\mathbf{\Theta})}{\partial\gamma} = \frac{n}{\gamma} - 2\sum_{i=1}^{n}\frac{\lambda^{\alpha}\left[\lambda+x_i^{\beta}\right]^{-\alpha}}{1-(1-\gamma)\lambda^{\alpha}\left[\lambda+x_i^{\beta}\right]^{-\alpha}}.$$

The maximum likelihood estimators of MOPLx distribution, $\hat{\mathbf{\Theta}}_{MLE} = (\hat{\alpha}_{MLE}, \hat{\beta}_{MLE}, \hat{\lambda}_{MLE}, \hat{\gamma}_{MLE})^T$, do not have an explicit form. Thus, maximum likelihood estimates can be solved by numerical technique using R language (R Core Team, 2020).

The mininmum Anderson-Darling estimator

The MADE is one of the minimum distance estimators based on an Anderson-Darling statistics. This class of estimators is based on minimizing any estimates of the cdf with the empirical distribution function (Dey et al., 2019). The Anderson-Darling distance function (ADF) is written as

$$A(\mathbf{\theta}) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log\left(F\left(X_{i:n}\,|\,\mathbf{\theta}\right)\right) + \log\left(\bar{F}\left(X_{n+1-i:n}\,|\,\mathbf{\theta}\right)\right)\right\},$$

where $\mathbf{\theta}$ is the vector of parameters in the cdf and $\bar{F}$ is $1-F$. The point estimation of MADE (Rodrigues, 2016) is defined as

$$A(\hat{\mathbf{\theta}}) = \min_{\mathbf{\theta}} A(\mathbf{\theta}).$$

Theomrem 5. Let $X_1, X_2, \ldots, X_n$ be an ordered iid random sample of size n from the MOPLx population with cdf shown in equation (2) and let $\mathbf{\Theta} = (\alpha, \beta, \lambda, \gamma)^T$ be the vector of the parameters. Then, the ADF of MOPLx distribution is

$$A(\boldsymbol{\Theta}) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log\left(\frac{1-\lambda^{\alpha}\left(\lambda + x_{(i)}^{\beta}\right)^{-\alpha}}{1-(1-\gamma)\left(\lambda^{\alpha}\left(\lambda + x_{(i)}^{\beta}\right)^{-\alpha}\right)}\right)\right.$$
$$\left. +\log\left(1-\frac{1-\lambda^{\alpha}\left(\lambda + x_{(n+1-i)}^{\beta}\right)^{-\alpha}}{1-(1-\gamma)\left(\lambda^{\alpha}\left(\lambda + x_{(n+1-i)}^{\beta}\right)^{-\alpha}\right)}\right)\right\}.$$

According to *Theorem 5,* its associated nonlinear equations are

$$\frac{\partial A(\boldsymbol{\Theta})}{\partial \alpha} = -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\frac{\boldsymbol{\Theta}_{\alpha}\left(X_{(i)}\,|\,\boldsymbol{\Theta}\right)}{F\left(X_{(i)}\,|\,\boldsymbol{\Theta}\right)} - \frac{\boldsymbol{\Theta}_{\alpha}\left(X_{(n+1-i)}\,|\,\boldsymbol{\Theta}\right)}{\bar{F}\left(X_{(n+1-i)}\,|\,\boldsymbol{\Theta}\right)}\right\} = 0, \tag{6}$$

$$\frac{\partial A(\boldsymbol{\Theta})}{\partial \beta} = -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\frac{\boldsymbol{\Theta}_{\beta}\left(X_{(i)}\,|\,\boldsymbol{\Theta}\right)}{F\left(X_{(i)}\,|\,\boldsymbol{\Theta}\right)} - \frac{\boldsymbol{\Theta}_{\beta}\left(X_{(n+1-i)}\,|\,\boldsymbol{\Theta}\right)}{\bar{F}\left(X_{(n+1-i)}\,|\,\boldsymbol{\Theta}\right)}\right\} = 0, \tag{7}$$

$$\frac{\partial A(\boldsymbol{\Theta})}{\partial \lambda} = -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\frac{\boldsymbol{\Theta}_{\lambda}\left(X_{(i)}\,|\,\boldsymbol{\Theta}\right)}{F\left(X_{(i)}\,|\,\boldsymbol{\Theta}\right)} - \frac{\boldsymbol{\Theta}_{\lambda}\left(X_{(n+1-i)}\,|\,\boldsymbol{\Theta}\right)}{\bar{F}\left(X_{(n+1-i)}\,|\,\boldsymbol{\Theta}\right)}\right\} = 0, \tag{8}$$

$$\frac{\partial A(\boldsymbol{\Theta})}{\partial \gamma} = -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\frac{\boldsymbol{\Theta}_{\gamma}\left(X_{(i)}\,|\,\boldsymbol{\Theta}\right)}{F\left(X_{(i)}\,|\,\boldsymbol{\Theta}\right)} - \frac{\boldsymbol{\Theta}_{\gamma}\left(X_{(n+1-i)}\,|\,\boldsymbol{\Theta}\right)}{\bar{F}\left(X_{(n+1-i)}\,|\,\boldsymbol{\Theta}\right)}\right\} = 0, \tag{9}$$

where $\boldsymbol{\Theta}_{\alpha}\left(X_{(i)}\,|\,\boldsymbol{\Theta}\right) = \dfrac{\gamma\lambda^{\alpha}\left(\lambda + x^{\beta}\right)^{-\alpha}\left[\ln\left(\lambda + x^{\beta}\right) - \ln\lambda\right]}{\left[1-(1-\gamma)\left(\lambda^{\alpha}\left(\lambda + x^{\beta}\right)^{-\alpha}\right)\right]^{2}},$

$$\boldsymbol{\Theta}_{\beta}\left(X_{(i)}\,|\,\boldsymbol{\Theta}\right) = \frac{\alpha\gamma\lambda^{\alpha}x^{\beta}(\ln x)\left(\lambda + x^{\beta}\right)^{-\alpha-1}}{\left[1-(1-\gamma)\left(\lambda^{\alpha}\left(\lambda + x^{\beta}\right)^{-\alpha}\right)\right]^{2}},$$

$$\boldsymbol{\Theta}_{\lambda}\left(X_{(i)}\,|\,\boldsymbol{\Theta}\right) = \frac{-\alpha\gamma x^{\beta}\lambda^{2\alpha-1}\left(\lambda + x^{\beta}\right)^{-2\alpha-1}}{\left[1-(1-\gamma)\left(\lambda^{\alpha}\left(\lambda + x^{\beta}\right)^{-\alpha}\right)\right]^{2}},$$

$$\boldsymbol{\Theta}_{\gamma}\left(X_{(i)}\,|\,\boldsymbol{\Theta}\right) = \frac{\lambda^{\alpha}\left(\lambda + x^{\beta}\right)^{-\alpha}\left(\lambda^{\alpha}\left(\lambda + x^{\beta}\right)^{-\alpha} - 1\right)}{\left[1-(1-\gamma)\left(\lambda^{\alpha}\left(\lambda + x^{\beta}\right)^{-\alpha}\right)\right]^{2}}.$$

Thus, $\hat{\boldsymbol{\Theta}}_{MADE} = (\hat{\alpha}_{MADE}, \hat{\beta}_{MADE}, \hat{\lambda}_{MADE}, \hat{\gamma}_{MADE})^{T}$, the Anderson-Darling estimated parameter vector, can be solved by equations (6)-(9). These equations can be obtained by numerically solving the system of non-linear equations through the R language (R Core Team, 2020).

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Results and discussion

1. Simulation study

In this part, the Monte Carlo simulation is applied to illustrate the comparison of the parameter estimation methods, MLE and MADE. All cases are selected based on the shape of the MOPLx distribution, unimodal and decreasing shapes, presented in Table 1. Each of the cases is constructed from MOPLx distribution with the sample sizes of 15, 30, 50, 100, 300, 500, 1000. The simulation is repeated 1,000 times and the parameter estimates for both methods are obtained by using Nelder-Mead algorithm, with *optim* function in *stats* package of R language (R Core Team, 2020). This function provides basic optimization capabilities, most widely used functions in R language (Nash, 2014). Moreover, the criteria for comparing the performance of MLE and MADE are bias, standard diviation (SD) and root mean square error (RMSE). Suppose $\hat{\mathbf{\Theta}}$ is the estimated parameter vector and $\mathbf{\Theta}$ is the true parameter vector. Then, the bias, SD and RMSE are computed by the fomulas: $\hat{\mathbf{\Theta}} = \sum_{i=1}^{1000} \hat{\mathbf{\Theta}}_i / 1000$, $\mathrm{Bias}(\hat{\mathbf{\Theta}}) = \hat{\mathbf{\Theta}} - \mathbf{\Theta}$, $\mathrm{SD}(\hat{\mathbf{\Theta}}) = \sqrt{\sum_{i=1}^{1000} (\hat{\mathbf{\Theta}}_i - \hat{\mathbf{\Theta}})^2 / 999}$, and $\mathrm{RMSE}(\hat{\mathbf{\Theta}}) = \sqrt{\sum_{i=1}^{1000} (\hat{\mathbf{\Theta}}_i - \mathbf{\Theta})^2 / 1000}$, respectively.

Table 1 provides all cases including mean and variance of the specified distributions. The characteristics of MOPLx shape in all cases are also illustrated in Figure 2 which has unimodal and decreasing shapes.

Table 1 All cases of the MOPLx parameters for simulation study and corresponding mean and variance

cases	**Parameter**				**Mean**	**Variance**
	α	β	λ	γ		
1	4.5	1.0	1.5	1.5	0.5353	0.4339
2	5.5	1.0	2.5	2.5	0.8925	0.8464
3	1.5	1.5	1.5	1.5	1.8658	15.2071
4	3.5	1.5	2.5	1.5	1.0217	0.6993
5	3.5	1.5	3.5	3.5	1.7718	1.6086

Random variate generation in the MOPLx distribution

The inversion method is utilized to generate a random variable for the MOPLx distribution by setting $x_i = F^{-1}(u_i)$, where u_i is a random variable of the uniform distribution on (0,1), which is denoted as $\mathrm{U}(0,1)$.

Let $X \sim \mathrm{MOPLx}(\alpha, \beta, \lambda, \gamma)$, the random variate generation of the MOPLx distribution can be constructed as following steps.

1) Denoting the true parameter values $(\alpha, \beta, \lambda, \gamma)$ and sample sizes (n)
2) Generating u_i from $\mathrm{U}(0,1)$, when $i = 1, 2, 3, \ldots, n$
3) Generating x_i from the equation (3)

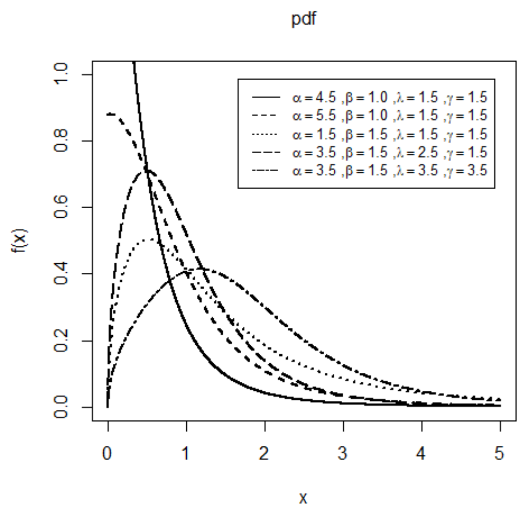

Figure 2 Plots of MOPLx distribution used in simulation study

Simulation study result

The results in simulation study based on various combination of parameter values and sample sizes are presented in Tables 2. Performance of these estimators is evaluated through their biases, SDs and RMSEs for both parameter estimation methods. It is shown that in most cases the MADE can produce lower bias, SD and RMSE values for the parameters α, β and λ when $n \le 50$. In case 3, MADE shows lower bias for all parameters when $n \le 50$. Almost all cases, the results indicate that SD and RMSE values of MADE are larger than those of MLE when $n \ge 100$. When considering the parameter β of two estimation methods, the estimated values of β are closer the true value than other parameters in all situations. Besides, in most cases, the bias, SD and RMSE values of the parameter γ using MADE seem to be higher than those using the MLE. This can be seen more clearly when $n \ge 500$. Furthermore, the results show that SD and RMSE values for MLE and MADE tend to decrease as the sample size increases in most situations. In addition, the values of the estimated parameters tend to be closer to the true parameters when n is large, except for the parameter λ with MADE in case 5. On overall result, we observe that the MLE performs quite well when n is large. However, MADE performs slightly better than MLE when n is small.

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Table 2 The estimates, Bias, SD and RMSE of the estimated parameters using MLE and MADE methods

Case 1: True parameters: $(\alpha = 4.5,\ \beta = 1.0,\ \lambda = 1.5,\ \gamma = 1.5)$

n		MLE				MADE			
		Estimate	Bias	SD	RMSE	Estimate	Bias	SD	RMSE
15	$\hat{\alpha}$	7.5944	3.0944	8.8426	9.3642	4.7193	0.2193	6.7944	**6.7946**
	$\hat{\beta}$	1.3177	0.3177	0.5238	0.6124	1.2180	0.2180	0.5539	**0.5950**
	$\hat{\lambda}$	6.7224	5.2224	9.0465	10.4418	1.9610	0.4610	3.0413	**3.0746**
	$\hat{\gamma}$	3.2406	1.7406	6.1328	**6.3721**	5.2051	3.7051	7.8878	8.7111
30	$\hat{\alpha}$	6.8482	2.3482	6.3254	6.7442	5.3211	0.8211	6.3656	**6.4152**
	$\hat{\beta}$	1.1474	0.1474	0.3538	**0.3831**	1.0807	0.0807	0.3977	0.4056
	$\hat{\lambda}$	4.8988	3.3988	6.3390	7.1899	1.8933	0.3933	2.4388	**2.4691**
	$\hat{\gamma}$	3.4493	1.9493	5.2029	**5.5536**	5.0377	3.5377	7.1726	7.9944
50	$\hat{\alpha}$	5.4098	0.9098	3.9864	**4.0871**	5.4985	0.9985	5.5545	5.6408
	$\hat{\beta}$	1.0958	0.0958	0.2820	**0.2977**	1.0259	0.0259	0.3234	0.3243
	$\hat{\lambda}$	2.5899	1.0899	2.4875	2.7147	1.8717	0.3717	2.3763	**2.4040**
	$\hat{\gamma}$	2.4802	0.9802	2.5300	**2.7121**	4.1478	2.6478	5.6217	6.2115
100	$\hat{\alpha}$	5.1307	0.6307	2.9617	**3.0266**	4.8309	0.3309	3.6947	3.7076
	$\hat{\beta}$	1.0442	0.0442	0.1975	**0.2023**	1.0065	0.0065	0.2278	0.2278
	$\hat{\lambda}$	2.2344	0.7344	2.0737	2.1989	1.6157	0.1157	1.7732	**1.7761**
	$\hat{\gamma}$	2.3617	0.8617	2.2116	**2.3726**	3.0764	1.5764	3.6924	4.0132
300	$\hat{\alpha}$	4.8328	0.3328	1.7940	**1.8237**	4.6434	0.1434	2.4906	2.4934
	$\hat{\beta}$	1.0109	0.0109	0.1219	**0.1224**	0.9872	-0.0128	0.1436	0.1441
	$\hat{\lambda}$	1.7533	0.2533	1.0543	**1.0838**	1.5354	0.0354	1.4130	1.4127
	$\hat{\gamma}$	1.8955	0.3955	1.1366	**1.2029**	2.3435	0.8435	1.7840	1.9726
500	$\hat{\alpha}$	4.5422	0.0422	1.3003	**1.3003**	4.4128	-0.0872	1.7926	1.7938
	$\hat{\beta}$	1.0204	0.0204	0.0895	**0.0918**	0.9918	-0.0082	0.1124	0.1126
	$\hat{\lambda}$	1.5868	0.0868	0.6404	**0.6460**	1.4098	-0.0902	0.8853	0.8894
	$\hat{\gamma}$	1.5935	0.0935	0.6435	**0.6500**	2.1190	0.6190	1.3390	1.4746
1000	$\hat{\alpha}$	4.4890	-0.0110	0.8999	**0.8995**	4.3176	-0.1824	1.7224	1.7312
	$\hat{\beta}$	1.0129	0.0129	0.0689	**0.0701**	0.9956	-0.0044	0.0793	0.0794
	$\hat{\lambda}$	1.5724	0.0724	0.5706	**0.5749**	1.4071	-0.0929	1.0033	1.0071
	$\hat{\gamma}$	1.5777	0.0777	0.5949	**0.5997**	1.9066	0.4066	0.9131	0.9991

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Table 2 (Continued)

Case 2: True parameters: $(\alpha = 5.5, \beta = 1.0, \lambda = 2.5, \gamma = 2.5)$

n		MLE				MADE			
		Estimate	Bias	SD	RMSE	Estimate	Bias	SD	RMSE
15	$\hat{\alpha}$	7.2916	1.7916	9.6804	9.8400	4.9582	-0.5418	7.4263	**7.4423**
	$\hat{\beta}$	1.5414	0.5414	0.7140	0.8957	1.4164	0.4164	0.6961	**0.8108**
	$\hat{\lambda}$	8.9950	6.495	12.7946	14.343	2.6025	0.1025	3.3924	**3.3923**
	$\hat{\gamma}$	3.8857	1.3857	7.4788	**7.6024**	5.8012	3.3012	8.5442	9.1558
30	$\hat{\alpha}$	7.0873	1.5873	7.3887	7.5536	5.8664	0.3664	7.0570	**7.0630**
	$\hat{\beta}$	1.2623	0.2623	0.4183	0.4936	1.1765	0.1765	0.4581	**0.4907**
	$\hat{\lambda}$	6.9371	4.4371	8.7655	9.8207	2.6416	0.1416	3.0915	**3.0932**
	$\hat{\gamma}$	4.2408	1.7408	6.4256	**6.6541**	5.9664	3.4664	7.8661	8.5924
50	$\hat{\alpha}$	7.0178	1.5178	6.0017	**6.1877**	6.5706	1.0706	6.9308	7.0095
	$\hat{\beta}$	1.1480	0.1480	0.3340	**0.3652**	1.0905	0.0905	0.3739	0.3845
	$\hat{\lambda}$	5.6067	3.1067	6.5691	7.2637	2.8021	0.3021	2.7958	**2.8107**
	$\hat{\gamma}$	4.5224	2.0224	5.8237	**6.1621**	5.8637	3.3637	7.4293	8.1520
100	$\hat{\alpha}$	6.2213	0.7213	3.8802	**3.9448**	6.2977	0.7977	5.7084	5.7611
	$\hat{\beta}$	1.0747	0.0747	0.2347	**0.2462**	1.0263	0.0263	0.2659	0.2670
	$\hat{\lambda}$	3.8638	1.3638	3.8040	4.0392	2.6973	0.1973	2.6605	**2.6665**
	$\hat{\gamma}$	3.8990	1.3990	4.0738	**4.3054**	5.0381	2.5381	5.8604	6.3837
300	$\hat{\alpha}$	6.2229	0.7229	2.7033	**2.7970**	6.1187	0.6187	4.0644	4.1092
	$\hat{\beta}$	1.0218	0.0218	0.1493	**0.1508**	0.9852	-0.0148	0.1721	0.1727
	$\hat{\lambda}$	3.2502	0.7502	2.0966	**2.2258**	2.6344	0.1344	2.1800	2.1830
	$\hat{\gamma}$	3.1250	0.6250	2.1749	**2.2619**	4.0783	1.5783	3.5080	3.8451
500	$\hat{\alpha}$	5.5618	0.0618	1.7673	**1.7675**	5.7079	0.2079	3.0642	3.0697
	$\hat{\beta}$	1.0275	0.0275	0.1078	**0.1112**	0.9874	-0.0126	0.1365	0.1370
	$\hat{\lambda}$	2.6777	0.1777	1.0744	**1.0885**	2.4458	-0.0542	1.6693	1.6693
	$\hat{\gamma}$	2.6178	0.1178	1.0782	**1.0841**	3.5367	1.0367	2.1785	2.4116
1000	$\hat{\alpha}$	5.5304	0.0304	1.2787	**1.2784**	5.2640	-0.2360	1.8118	1.8262
	$\hat{\beta}$	1.0196	0.0196	0.0801	**0.0824**	0.9844	-0.0156	0.1028	0.1040
	$\hat{\lambda}$	2.6520	0.1520	0.8465	**0.8596**	2.2350	-0.2650	1.2279	1.2555
	$\hat{\gamma}$	2.5124	0.0124	0.8508	**0.8505**	3.3942	0.8942	1.8499	2.0538

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Table 2 (Continued)

Case 3: True parameters: $(\alpha = 1.5,\ \beta = 1.5,\ \lambda = 1.5,\ \gamma = 1.5)$

n		MLE				MADE			
		Estimate	**Bias**	**SD**	**RMSE**	**Estimate**	**Bias**	**SD**	**RMSE**
15	$\hat{\alpha}$	2.2996	0.7996	2.9252	3.0311	2.0899	0.5899	2.9654	**3.0221**
	$\hat{\beta}$	2.1176	0.6176	1.2179	1.3650	1.8937	0.3937	0.9826	**1.0581**
	$\hat{\lambda}$	2.7815	1.2815	3.3474	3.5828	2.2776	0.7776	2.8247	**2.9284**
	$\hat{\gamma}$	2.8544	1.3544	3.1850	**3.4596**	2.4373	0.9373	3.3655	3.4919
30	$\hat{\alpha}$	2.0326	0.5326	1.8543	**1.9284**	1.9557	0.4557	1.9049	1.9578
	$\hat{\beta}$	1.7862	0.2862	0.7666	0.8179	1.6659	0.1659	0.6345	**0.6555**
	$\hat{\lambda}$	2.6640	1.1640	2.7645	2.9983	2.0624	0.5624	2.2332	**2.3019**
	$\hat{\gamma}$	2.4603	0.9603	2.3859	**2.5708**	2.3684	0.8684	2.5291	2.6728
50	$\hat{\alpha}$	1.8921	0.3921	1.3103	**1.3670**	1.8612	0.3612	1.6847	1.7222
	$\hat{\beta}$	1.6356	0.1356	0.5272	0.5441	1.5936	0.0936	0.4811	**0.4899**
	$\hat{\lambda}$	2.4747	0.9747	2.2718	2.4710	1.9865	0.4865	2.0086	**2.0657**
	$\hat{\gamma}$	2.2961	0.7961	2.2269	2.3638	2.2257	0.7257	2.1296	**2.2488**
100	$\hat{\alpha}$	1.6463	0.1463	0.8204	**0.8330**	1.7097	0.2097	1.0869	1.1064
	$\hat{\beta}$	1.5737	0.0737	0.2990	0.3078	1.5329	0.0329	0.3004	**0.3021**
	$\hat{\lambda}$	1.9163	0.4163	1.3892	**1.4495**	1.9414	0.4414	1.8593	1.9101
	$\hat{\gamma}$	1.9016	0.4016	1.3257	**1.3846**	2.1258	0.6258	2.0421	2.1348
300	$\hat{\alpha}$	1.5657	0.0657	0.4015	**0.4066**	1.6093	0.1093	0.5197	0.5308
	$\hat{\beta}$	1.5209	0.0209	0.1536	**0.1549**	1.5052	0.0052	0.1731	0.1731
	$\hat{\lambda}$	1.8725	0.3725	1.0695	**1.1320**	1.8464	0.3464	1.4864	1.5255
	$\hat{\gamma}$	1.7020	0.2020	1.0635	**1.0820**	1.9023	0.4023	1.4949	1.5473
500	$\hat{\alpha}$	1.5454	0.0454	0.3112	**0.3143**	1.5581	0.0581	0.3505	0.3551
	$\hat{\beta}$	1.5080	0.0080	0.1238	**0.1240**	1.5035	0.0035	0.1343	0.1343
	$\hat{\lambda}$	1.7987	0.2987	1.0090	**1.0518**	1.7825	0.2825	1.1767	1.2096
	$\hat{\gamma}$	1.7237	0.2237	1.0177	**1.0416**	1.7569	0.2569	1.1594	1.1869
1000	$\hat{\alpha}$	1.5257	0.0257	0.2123	**0.2137**	1.5266	0.0266	0.2371	0.2385
	$\hat{\beta}$	1.5047	0.0047	0.0852	**0.0853**	1.5025	0.0025	0.0916	0.0915
	$\hat{\lambda}$	1.6834	0.1834	0.6602	**0.6849**	1.6922	0.1922	0.8577	0.8786
	$\hat{\gamma}$	1.5775	0.0775	0.6529	**0.6571**	1.6430	0.1430	0.8700	0.8812

Table 2 (Continued)

Case 4: True parameters: $(\alpha = 3.5, \beta = 1.5, \lambda = 2.5, \gamma = 1.5)$

n		MLE				MADE			
		Estimate	**Bias**	**SD**	**RMSE**	**Estimate**	**Bias**	**SD**	**RMSE**
15	$\hat{\alpha}$	4.0397	0.5397	5.8038	5.8259	3.1707	-0.3293	5.4185	**5.4258**
	$\hat{\beta}$	2.2772	0.7772	1.1371	1.3769	2.0776	0.5776	0.9631	**1.1226**
	$\hat{\lambda}$	5.688	3.1880	8.4165	8.9961	2.0365	-0.4635	2.1994	**2.2466**
	$\hat{\gamma}$	3.5916	2.0916	6.3567	**6.6889**	4.4852	2.9852	7.3746	7.9525
30	$\hat{\alpha}$	4.3408	0.8408	4.1230	**4.2063**	4.5873	1.0873	6.3846	6.4734
	$\hat{\beta}$	1.7572	0.2572	0.5900	**0.6434**	1.6487	0.1487	0.6284	0.6454
	$\hat{\lambda}$	5.1532	2.6532	5.9619	6.5236	2.4527	-0.0473	2.5674	**2.5666**
	$\hat{\gamma}$	3.5434	2.0434	5.0724	**5.4667**	5.2100	3.7100	7.3954	8.2706
50	$\hat{\alpha}$	3.9360	0.4360	2.8054	**2.8376**	4.2011	0.7011	4.4836	4.5359
	$\hat{\beta}$	1.6466	0.1466	0.4575	**0.4802**	1.548	0.0480	0.5087	0.5107
	$\hat{\lambda}$	4.0575	1.5575	3.9521	4.2461	2.3704	-0.1296	2.1762	**2.1790**
	$\hat{\gamma}$	2.9433	1.4433	3.4037	**3.6955**	4.6024	3.1024	6.3377	7.0535
100	$\hat{\alpha}$	3.8484	0.3484	1.9935	**2.0227**	4.1412	0.6412	3.7529	3.8054
	$\hat{\beta}$	1.5431	0.0431	0.2930	**0.2960**	1.4830	-0.0170	0.3380	0.3382
	$\hat{\lambda}$	3.4319	0.9319	2.9586	3.1005	2.5535	0.0535	2.4970	**2.4963**
	$\hat{\gamma}$	2.6822	1.1822	2.6190	**2.8723**	3.3881	1.8881	4.0302	4.4487
300	$\hat{\alpha}$	3.7484	0.2484	1.3284	**1.3508**	3.8532	0.3532	2.1359	2.1639
	$\hat{\beta}$	1.4954	-0.0046	0.1821	**0.1821**	1.4534	-0.0466	0.2083	0.2134
	$\hat{\lambda}$	2.7548	0.2548	1.5019	**1.5226**	2.4769	-0.0231	1.9653	1.9644
	$\hat{\gamma}$	2.0616	0.5616	1.2737	**1.3914**	2.5187	1.0187	1.8723	2.1307
500	$\hat{\alpha}$	3.5808	0.0808	0.9539	**0.9568**	3.5939	0.0939	1.3700	1.3725
	$\hat{\beta}$	1.5254	0.0254	0.1322	**0.1346**	1.4781	-0.0219	0.1649	0.1663
	$\hat{\lambda}$	2.7558	0.2558	1.0874	**1.1166**	2.4406	-0.0594	1.4444	1.4449
	$\hat{\gamma}$	1.5795	0.0795	0.6451	**0.6497**	2.0793	0.5793	1.2468	1.3742
1000	$\hat{\alpha}$	3.5214	0.0214	0.6247	**0.6248**	3.417	-0.0830	0.7067	0.7112
	$\hat{\beta}$	1.5159	0.0159	0.0981	**0.0993**	1.4854	-0.0146	0.1154	0.1163
	$\hat{\lambda}$	2.6958	0.1958	0.9645	**0.9837**	2.3354	-0.1646	1.0068	1.0196
	$\hat{\gamma}$	1.5650	0.0650	0.5888	**0.5921**	1.8937	0.3937	0.8637	0.9488

Table 2 (Continued)

Case 4: True parameters: $(\alpha = 3.5, \beta = 1.5, \lambda = 2.5, \gamma = 1.5)$

n		MLE				MADE			
		Estimate	**Bias**	**SD**	**RMSE**	**Estimate**	**Bias**	**SD**	**RMSE**
15	$\hat{\alpha}$	3.8433	0.3433	6.4688	6.4747	3.3691	-0.1309	5.3215	**5.3205**
	$\hat{\beta}$	3.5819	2.0819	4.7186	5.1553	2.5687	1.0687	2.1115	**2.3656**
	$\hat{\lambda}$	7.3439	3.8439	9.9849	10.6946	3.4776	-0.0224	4.1414	**4.1394**
	$\hat{\gamma}$	5.6286	2.1286	7.9100	8.1876	5.7790	2.2790	7.2747	**7.6199**
30	$\hat{\alpha}$	3.4760	-0.0240	3.4186	**3.4170**	3.5918	0.0918	4.4057	4.4044
	$\hat{\beta}$	2.1437	0.6437	1.2488	1.4044	1.9661	0.4661	0.9659	**1.0721**
	$\hat{\lambda}$	6.4061	2.9061	6.8385	7.4272	3.5120	0.0120	3.7118	**3.7099**
	$\hat{\gamma}$	5.1620	1.6620	6.3467	6.5577	5.4061	1.9061	5.8363	**6.1369**
50	$\hat{\alpha}$	3.6702	0.1702	2.8448	**2.8485**	4.0017	0.5017	4.2944	4.3214
	$\hat{\beta}$	1.8232	0.3232	0.6993	0.7701	1.7407	0.2407	0.5744	**0.6225**
	$\hat{\lambda}$	5.6645	2.1645	4.7163	5.1871	3.8991	0.3991	3.6530	**3.6729**
	$\hat{\gamma}$	4.4323	0.9323	4.4795	**4.5734**	5.1143	1.6143	5.2156	5.4572
100	$\hat{\alpha}$	3.5492	0.0492	1.9031	**1.9028**	4.1110	0.6110	3.1583	3.2153
	$\hat{\beta}$	1.6570	0.1570	0.3428	**0.3769**	1.6083	0.1083	0.3909	0.4054
	$\hat{\lambda}$	4.3993	0.8993	2.4269	**2.5870**	4.4074	0.9074	3.6409	3.7505
	$\hat{\gamma}$	3.6554	0.1554	2.3673	**2.3712**	4.5631	1.0631	4.1448	4.2770
300	$\hat{\alpha}$	3.6060	0.1060	1.1898	**1.1939**	3.9563	0.4563	2.1618	2.2084
	$\hat{\beta}$	1.5546	0.0546	0.1999	**0.2071**	1.5421	0.0421	0.2261	0.2298
	$\hat{\lambda}$	4.0568	0.5568	1.7041	**1.7920**	4.5201	1.0201	3.4479	3.5940
	$\hat{\gamma}$	3.6593	0.1593	1.7986	**1.8047**	3.9176	0.4176	2.7284	2.7588
500	$\hat{\alpha}$	3.4305	-0.0695	0.8323	**0.8348**	3.7246	0.2246	1.5108	1.5267
	$\hat{\beta}$	1.5533	0.0533	0.1544	**0.1633**	1.5300	0.0300	0.1896	0.1918
	$\hat{\lambda}$	3.7450	0.2450	1.1397	**1.1651**	4.1986	0.6986	2.8203	2.9041
	$\hat{\gamma}$	3.4301	-0.0699	1.2353	**1.2366**	3.9309	0.4309	2.5186	2.5540
1000	$\hat{\alpha}$	3.4964	-0.0036	0.6532	**0.6529**	3.6609	0.1609	0.9736	0.9863
	$\hat{\beta}$	1.5235	0.0235	0.1178	**0.1201**	1.5188	0.0188	0.1455	0.1467
	$\hat{\lambda}$	3.6809	0.1809	0.9917	**1.0076**	4.1661	0.6661	2.4035	2.4929
	$\hat{\gamma}$	3.5142	0.0142	1.0645	**1.0641**	3.8216	0.3216	2.4397	2.4597

Note: Bold values denote the smaller RMSE of the estimated parameters between MLE and MADE methods.

2. Applications

For applications, we implement the parameter estimation methods on four real data sets. The data sets are considered to fit with MOPLx distribution and their parameter estimations are carried out using MLE and MADE. Then, the fitted models are compared using goodness - of - fit tests based on Komogorov Smirnov (K-S) and Anderson-Darling (A-D), reported in Table 3. The lower values of these statistics and higher *p*-values of K-S and A-D indicate good fits. In addition, the histograms of all data sets and plots of the estimated pdf and cdf of the MOPLx distribution are illustrated in Figure 3.

The first data is corresponding to the tensile strength of 100 carbon fibers in GPa (Data I). The data has been obtained from Nichols & Padgett (2006). The values of mean $(\bar{x})$, SD, skewness, and kurtosis of the tensile strength of carbon fibers are 2.6014, 1.0209, 0.3590, and 3.1262, respectively. The data set is given as follows: 3.70, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 0.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65.

The second data is based on the waiting times between 65 consecutive eruptions in seconds of the Kiama Blowhole in Australia (Data II). The times occurred from 1340 hours on 12 July 1998 were observed using a digital watch (Pinho et al., 2015). The summary statistics for this data set are $\bar{x}$ = 39.8281, SD = 33.7505, skewness = 1.5464, and kurtosis = 5.7711. The data set is reported as follows: 83, 51, 87, 60, 28, 95, 8, 27, 15, 10, 18, 16, 29, 54, 91, 8, 17, 55, 10, 35,47, 77, 36, 17, 21, 36, 18, 40, 10, 7, 34, 27, 28, 56, 8, 25, 68, 146, 89, 18, 73, 69, 9, 37, 10, 82, 29, 8, 60, 61, 61, 18, 169, 25, 8, 26, 11, 83, 11, 42, 17, 14, 9, 12.

The third data set was considered by Bhaumik et al. (2009) for small sample size. The data set consists of 34 observations on the vinyl chloride data which is obtained from clean upgrading monitoring wells in mg/L (Data III). The data set is given as follows: 5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2. The values of $\bar{x}$, SD, skewness, and kurtosis of the the vinyl chloride data are 1.8794, 1.9526, 1.6037, and 5.0054, respectively.

The fourth data set is obtained from Jamal et al. (2018) and it represents 30 successive values of March precipitation (inches) in Minneapolis/St Paul (Data IV). The data set is reported as follows: 0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05. The summary statistics for the last data set are $\bar{x}$ = 1.6750, SD = 1.0006, skewness = 1.0867, and kurtosis = 4.2069.

Table 3 shows the estimated values of parameters in the MOPLx distribution with K-S and A-D test statistic along with *p*-values on four real data sets. Based on the results in Table 3, the efficiency of MLE and MADE methods and fitting performance of the MOPLx distribution can be evaluate. First, the results show that the MOPLx distribution is appropriate in describing four real data sets in terms of both K-S test and A-D test. Second, the MLE method is likely to present greater *p*-values in the K-S test and A-D test for both Data I and Data II. The first and second data sets represent large and moderate sample size with n= 100 and n= 65, respectively.

Table 3 The parameter estimated values and goodness-of-fit statistics for four data sets via the MLE and MADE methods

Data	Method	Estimates				K-S (*p*-value)	A-D (*p*-value)
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\gamma}$		
I	MLE	5.3856	1.7254	5.9759	26.4715	0.0603 (0.8609)	0.4708 (0.7763)
	MADE	3.4227	2.0359	4.9574	17.9661	0.0611 (0.8499)	0.5124 (0.7338)
II	MLE	0.8057	2.3020	45.7421	21.8750	0.09355 (0.6297)	0.1372 (0.1795)
	MADE	0.8078	1.5281	41.1176	2.4918	0.8099 (0.4734)	1.8146 (0.1166)
III	MLE	5.6308	1.2140	15.4653	0.5578	0.0773 (0.9872)	0.2063 (0.9886)
	MADE	2.9301	1.2083	6.8841	0.6234	0.0735 (0.9930)	0.1849 (0.9941)
IV	MLE	2.5404	2.2670	7.7842	0.9921	0.0628 (9.9978×10^{-1})	0.1171 (9.9987×10^{-1})
	MADE	2.8032	2.1460	8.6366	0.9552	0.0590 (9.9994×10^{-1})	1.8146 (9.9998×10^{-1})

Furthermore, Figure 3 shows the appropriateness of the fitted model for these data sets on the basis of histogram and density plots. The plots also confirm that MLE provides better fit because its curve is relatively close to the empirical density curve. Third, the *p*-values based on goodness - of - fit tests support that MADE method is slightly better than MLE in Data III and Data IV that represents small sample sizes. The third and fourth data sets consist of 34 and 30 observations, repectively. Additionally, Figure 4 illustrates that the MOPLx under the MADE method is closely fitted to these data sets.

Conclusion

In this paper, two estimation techniques, including maximum likelihood estimator (MLE) and minimum Anderson-Darling estimator (MADE) for estimating the unknown parameters of the Marshall-Olkin Power Lomax (MOPLx) distribution are studied. The MADE method can be implemented based on approximated cumulative distribution function and the MLE method can be applied based on approximated probability density function, as discussed in Raschke (2017). We also compare the performance of these methods for MOPLx distribution through a simulation study and real data sets. In simulation study, the MADE method is derived and compared its performance to the MLE via bias, SD and RMSE. The simulated study indicates that MLE performs better than MADE as sample sizes are large. However, the MADE method performs slightly better for smaller sample sizes than MLE. In some cases of MADE method, the bias, SD, and RMSE values are not much larger than that of the MLE method but the RMSE can be even smaller in most cases of small sample size. Then, the applications for the MOPLx distribution with four real data sets are demonstrated, which parameters are estimated using MLE and MADE. In this work, the measurements, K-S test and A-D test, are considered for assessment and comparison of parameter estimation methods. The results show that MLE

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method outperforms its competitor for estimating the parameter of the MOPLx in the first and second data sets, and MADE method seems to fit the third and fourth data sets slightly better than MLE method. Therefore, it is concluded from both simulated and real data sets that the two methods show identical performance for estimating the parameters of MOPLx distribution unless the sample size is small. However, the MADE performs slightly better for smaller sample sizes.

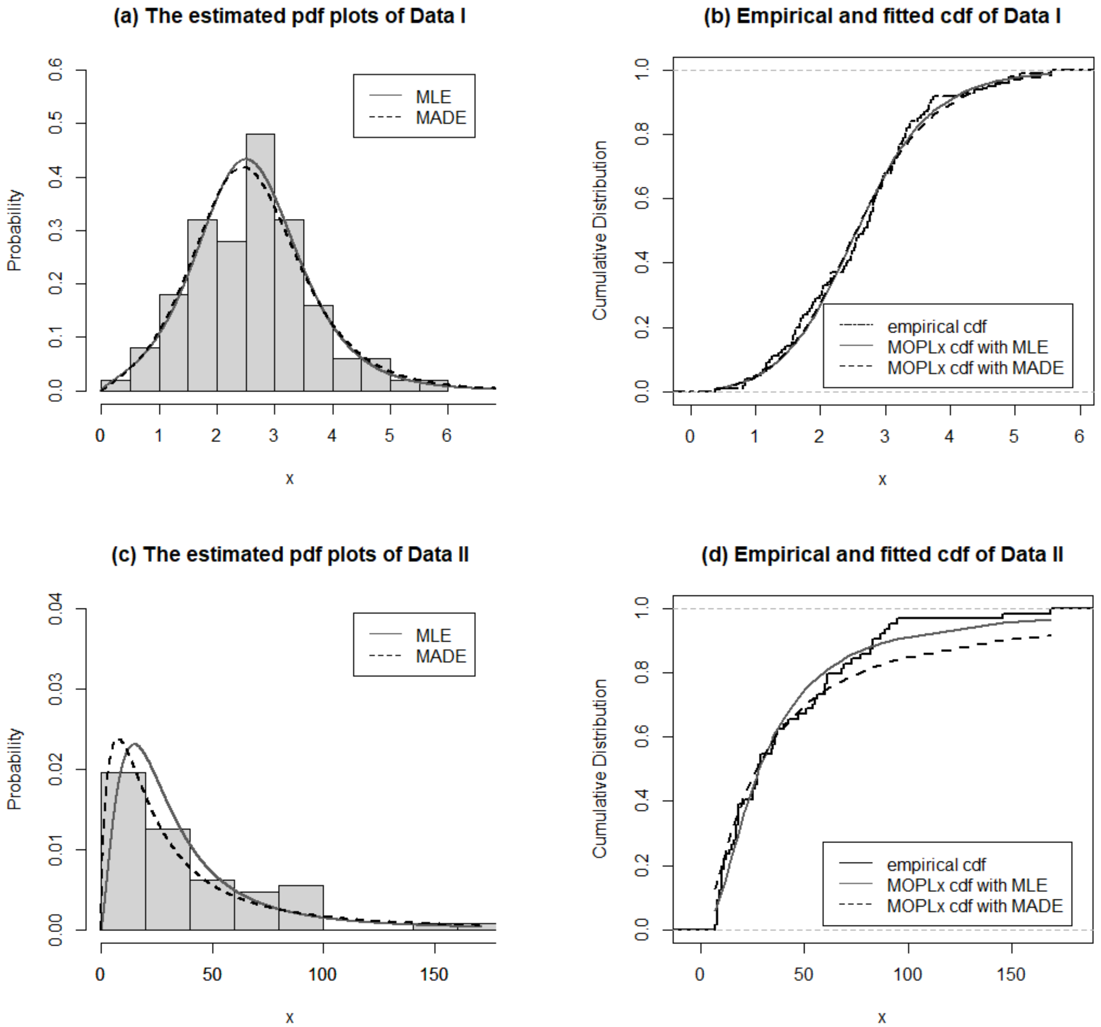

Figure 3 The histogram of data and estimated pdf, and the empirical and theoretical cdf plot of MOPLx distribution using MLE and MADE for Data I and Data II

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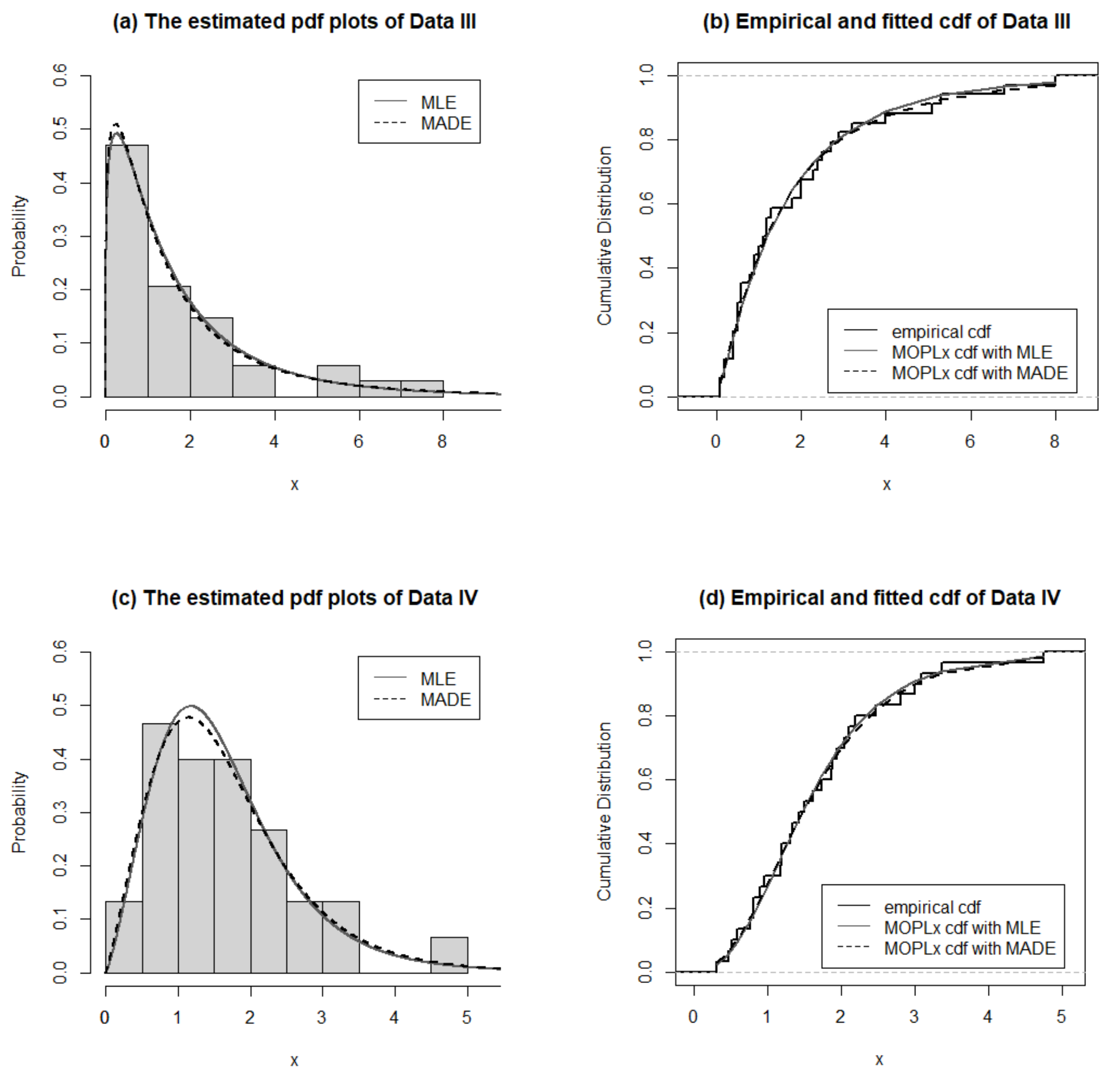

Figure 4 The histogram of data and estimated pdf, and the empirical and theoretical cdf plot of MOPLx distribution using MLE and MADE for Data III and Data IV

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