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Research Article

Estimating the Average Run Length of CUSUM Control Chart for Seasonal Autoregressive Integrated Moving Average of Order $(P,D,Q)_L$ Model

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Abstract

The main objective of this paper is to present the numerical integral equation of average run length (ARL) of cumulative sum (CUSUM) control chart when observations are seasonal autoregressive integrated moving average denoted by SARIMA$(P,D,Q)_L$ process with exponential white noise. The accuracy of the proposed method is established by comparing them to the Monte Carlo simulation method in term of the absolute percentage difference and the computational times. A comparative results obtained from the numerical integral equation with the Monte Carlo simulation shows that the absolute percentage difference is less than 1.0%. In terms of computational time, we find that the computational times are approximately 60 minutes for the numerical integration method and approximately 190 minutes for the Monte Carlo simulation method.

Keywords: Average Run Length, Cumulative Sum Control Chart, Seasonal Autoregressive Integrated Moving Average

Introduction

In general, the production process has a variety of steps in the production of products in order to get the good quality products. Statistical process control (SPC) is an important for the production process to control and improve the quality of production process. Control charts are one of the efficient tools of SPC for detecting changes in mean or variations of the process. Therefore, the objective of control chart consists in detecting variations of the process over time. There are many applications of control charts in engineering, medical, health-care, finance, chemistry and biology etc.

The concept of a control chart was first proposed by Walter A. Shewhart (1931), which is called the Shewhart control chart. It is a quality control chart that is suitable for detecting large changes in the process. Recently, the cumulative sum (CUSUM) control chart presented by Page (1954), which is effective in detecting small shifts in the process. In this paper, we discuss the application of the CUSUM control chart for monitoring in the mean of process. The CUSUM control chart are widely used in engineering and in health-care.

The adopted criterion of measures to confirm the performance of a control chart is the average run length (ARL). The most frequently used operating characteristics are in-control average run length or ARL_0 and out-of-control average run length or ARL_1. There are many methods for evaluating the ARL such as Markov Chain approach (MCA), integral equation approach (IE) and Monte Carlo simulations (MC).

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A systematic review demonstrated that, there are many researchers have been proposed for evaluating the ARL of CUSUM control chart such as Brook & Evans (1972) used the Markov Chain approach with finite state to evaluate the average run length of CUSUM control chart. Vanbrackle & Reynold (1997) used the integral equation method and the Markov Chain approach to evaluate the ARL of the EWMA and the CUSUM control charts for the autoregressive, AR(1) process with additional random error. Later, Areepong (2009) presented the formula for the ARL of the EWMA control chart when the data were exponential distribution by Martingale Approach and compared the efficiency of the formula for the ARL of the EWMA control chart with the CUSUM control chart. Later, Mititelu et al. (2010) developed an analytical solution for the ARL by using the Fredholm integral equation approach for CUSUM control chart when observations are hyperexponential distribution.

There are many situations in which the process is a serial correlation, thus these process need to be monitored by appropriate control charts. For this reason, several authors have been developed for evaluating the ARL of CUSUM control chart when the process is an autocorrelation, such as Busaba et al. (2012) investigated the explicit formula of the ARL for CUSUM control chart by using the Fredholm integral equation when observations are an autoregressive, AR(1) process. Later, Phanyaem et al. (2014) derived the analytical formula and used the numerical methods to find the ARL of CUSUM control chart for an autoregressive and moving average, ARMA(1,1) process with exponential white noise. In addition, Phanyaem et al. (2014) presented the numerical integral equation of ARL for an autoregressive and moving average process, ARMA(p,q) process with exponential white noise based on EWMA and CUSUM control charts and compared the ARL obtained from the EWMA control chart with the ARL obtained from the CUSUM control charts. Recently, Petcharat et al (2015) used the Fredholm integral equation approach to solve the exact formula of the ARL of CUSUM control chart based on a moving average, MA(q) process with exponential white noise. After that, Phanyaem (2017) proposed the explicit formula and the numerical integral equation to approximate the ARL of CUSUM control chart for $SARMA(1,1)_L$ process.

The objectives of this paper are to approximate the ARL of CUSUM control chart for a seasonal autoregressive integrated moving average, $SARIMA(P,D,Q)_L$ process with exponential distribution white noise by using the numerical integral equation method based on Gauss-Legendre quadrature rule and to compare the numerical results obtained from the numerical integral equation method with the results obtained from the Monte Carlo simulation method in term of the absolute percentage difference and the computational time.

The paper is organized as follows: in Section 1, we present the background and literature review of control charts. In Section 2, the underlying a seasonal autoregressive integrated moving average, CUSUM control chart based on $SARIMA(P,D,Q)_L$ process is introduced. In Section 3, the methodology of numerical integration is demonstrated using Gauss-Legendre quadrature rule. The Monte Carlo simulation technique is presented in Section 4. In Section 5, we compared the results from the numerical integral equation method with the results from the Monte Carlo simulation method. Finally, conclusion of this research are discussed.

CUSUM Control Chart based on $SARIMA(P,D,Q)_L$ Process

In this section, we present the procedure to generate the cumulative sum (CUSUM) control chart. The CUSUM control chart is used for the data that has subgroups or single observation at each time period. In addition, the CUSUM control chart can specify a fast initial response values to enhance the control chart. The CUSUM control chart displays the cumulative data of current and previous samples. For this reason, the CUSUM control chart is better than the Shewhart control chart for detecting small shifts in the mean of a process. In this paper, we used the CUSUM control chart to monitor the mean of a process based on SARIMA(P,D,Q)$_L$ process.

Let X_t is a seasonal autoregressive integrated moving average process with exponential white noise. The SARIMA process denoted by SARIMA(P,D,Q)$_L$ can be expressed as

$$(1-\Phi_{1,L}B^{L}-\Phi_{2,L}B^{2L}-...-\Phi_{P,L}B^{PL})(1-B^{L})^{D}X_t \ = \ \mu+(1-\Theta_{1,L}B^{L}-\Theta_{2,L}B^{2L}-...-\Theta_{Q,L}B^{QL})\xi_t \qquad (1)$$

where $\xi_t\ ; t = 1, 2, ...$ is the white noise process with exponential distribution, $\varPhi_P(\mathrm{B}^L)$ and $\varTheta_Q(\mathrm{B}^L)$ are the polynomials of order P and Q of the seasonal autoregressive and moving average components, respectively. The seasonal difference components are represented by $(1-B^{L})^{D}$.

The CUSUM control chart for monitoring the process mean is based on the statistics

$$C_t \ = \ max(C_{t-1}+X_t-a,\ 0)\ ;\ t=1,2,\ldots \qquad (2)$$

where X_t is the tth observation based on SARIMA(P,D,Q)$_L$ process and a is a reference value of CUSUM control chart.

The stopping time (τ_h) of CUSUM control chart is given by:

$$\tau_h \ = \inf\left\{t>0; C_t>h\right\},\ h>u \qquad (3)$$

where h is the upper control limit of CUSUM control chart.

Numerical Integration of ARL of CUSUM Chart for SARIMA(P,D,Q)$_L$

In this section we present the scheme to evaluate numerically the solutions of the integral equation by using Gauss-Legendre quadrature rule. Since the corresponding integral equation are difficult to solve, the numerical integral equation (NIE) method is the alternative way to solve the problem of integral equation. The numerical schemes are discussed by Kantorovich & Krylov (1960), Petrovskii (1996), Atkinson & Han (2001).

Suppose that the function $L(u)$ is the ARL of CUSUM control chart based on SARIMA(P,D,Q)$_L$ process with the initial value $C_0 = u$ where $u \in [0,h]$.

The ARL of CUSUM control chart is given by

$$ARL = L(u) = \mathbf{E}_{\infty}(\tau_h) < \infty. \qquad (4)$$

The solution of integral equation is given by the equation

$$L(u) = 1+\mathbf{E}_c\left[I\{0<C_1<h\}L(C_1)\right]+\mathbf{P}_c\{C_1=0\}L(0). \qquad (5)$$

where $\mathbf{P}_c$ is the probability measure

$\mathbf{E}_c$ is the expectation corresponding to an initial value *u*.

Therefore, an explicit formula for the solution of (5) is sought in the form of Fredholm's integral equation of the 2nd kind as follows

$$L(u) = 1 + \beta e^{\beta(u-a+\mu+(\Theta_L\mathrm{B}^L+...+\Theta_{QL}\mathrm{B}^{QL})\ \xi_t-\sum_{k=1}^{D}(-1)^{Lk}B^{Lk}X_t+(\Phi_L\mathrm{B}^L+...+\Phi_{PL}\mathrm{B}^{PL})(1-\mathrm{B}^L)^D X_t)}\int_0^h L(y)e^{-\beta y}dy$$

$$+\ (1-e^{-\beta(a-u-\mu-(\Theta_L\mathrm{B}^L+...+\Theta_{QL}\mathrm{B}^{QL})\ \xi_t+\sum_{k=1}^{D}(-1)^{Lk}B^{Lk}X_t-(\Phi_L\mathrm{B}^L+...+\Phi_{PL}\mathrm{B}^{PL})(1-\mathrm{B}^L)^D X_t)})L(0). \qquad (6)$$

where β is a parameter of exponential white noise.

Here, we assume that the $\tilde{L}(u)$ is the numerical integration solution of $L(u)$. This section, we consider the numerical solution of the ARL of CUSUM control chart governed by integral equations, which is described by

$$\tilde{L}(u) = 1 + L(0)F(a-u-\mu-(\Theta_L B^L+...+\Theta_{QL}B^{QL})\ \xi_t + \sum_{k=1}^{D}(-1)^{Lk}B^{Lk}X_t - (\Phi_L B^L+...+\Phi_{PL}B^{PL})(1-B^L)^D X_t)$$
$$+\int_0^h L(y)f(y+a-u-\mu-(\Theta_L B^L+...+\Theta_{QL}B^{QL})\ \xi_t + \sum_{k=1}^{D}(-1)^{Lk}B^{Lk}X_t - (\Phi_L B^L+...+\Phi_{PL}B^{PL})(1-B^L)^D X_t)dy \tag{7}$$

where

$$F(u) = 1-e^{-\beta u} \quad \text{and} \quad f(u) = \frac{dF(u)}{du} = \beta e^{-\beta u}.$$

The quadrature formula for approximating $\int_0^h f(y)dy$ will be h/m times the formula for approximating the equivalent integral over $[0,h]$ with the division points $0 \le a_1 \le ... \le a_m \le h$ and weights $w_j = h/m \ge 0$. Therefore, we can approximate the integral equation by a sum of areas of rectangles, which is described by

$$\int_0^b W(y)f(y)dy \approx \sum_{j=1}^{m} w_j f(a_j) \quad \text{with} \quad a_j = \frac{h}{m}\left(j-\frac{1}{2}\right) \quad ; j = 1,2,...,m.$$

Using the numerical integral equation (NIE) method. The numerical results of $L(u)$ is given by

$$\tilde{L}(a_i) = 1 + \tilde{L}(a_i)\, F(a-a_i-\mu-(\Theta_L B^L+...+\Theta_{QL}B^{QL})\ \xi_t + \sum_{k=1}^{D}(-1)^{Lk}B^{Lk}X_t - (\Phi_L B^L+...+\Phi_{PL}B^{PL})(1-B^L)^D X_t$$
$$+\sum_{j=1}^{m} w_j\, \tilde{L}(a_j)\ f(a_j+a-a_i-\mu-(\Theta_L B^L+...+\Theta_{QL}B^{QL})\ \xi_t + \sum_{k=1}^{D}(-1)^{Lk}B^{Lk}X_t - (\Phi_L B^L+...+\Phi_{PL}B^{PL})(1-B^L)^D X_t). \tag{8}$$

Consequently, we obtain the linear system as

$$\tilde{L}(a_1) = 1 + \tilde{L}(a_1)\, F(a-a_1-\mu-(\Theta_L B^L+...+\Theta_{QL}B^{QL})\ \xi_t + \sum_{k=1}^{D}(-1)^{Lk}B^{Lk}X_t - (\Phi_L B^L+...+\Phi_{PL}B^{PL})(1-B^L)^D X_t$$
$$+\sum_{j=1}^{m} w_j\, \tilde{L}(a_j)\ f(a_j+a-a_1-\mu-(\Theta_L B^L+...+\Theta_{QL}B^{QL})\ \xi_t + \sum_{k=1}^{D}(-1)^{Lk}B^{Lk}X_t - (\Phi_L B^L+...+\Phi_{PL}B^{PL})(1-B^L)^D X_t).$$

$$\tilde{L}(a_2) = 1 + \tilde{L}(a_2)\, F(a-a_2-\mu-(\Theta_L B^L+...+\Theta_{QL}B^{QL})\ \xi_t + \sum_{k=1}^{D}(-1)^{Lk}B^{Lk}X_t - (\Phi_L B^L+...+\Phi_{PL}B^{PL})(1-B^L)^D X_t$$
$$+\sum_{j=1}^{m} w_j\, \tilde{L}(a_j)\ f(a_j+a-a_2-\mu-(\Theta_L B^L+...+\Theta_{QL}B^{QL})\ \xi_t + \sum_{k=1}^{D}(-1)^{Lk}B^{Lk}X_t - (\Phi_L B^L+...+\Phi_{PL}B^{PL})(1-B^L)^D X_t).$$

$$\vdots$$

$$\tilde{L}(a_m) = 1 + \tilde{L}(a_m)\, F(a-a_m-\mu-(\Theta_L B^L+...+\Theta_{QL}B^{QL})\ \xi_t + \sum_{k=1}^{D}(-1)^{Lk}B^{Lk}X_t - (\Phi_L B^L+...+\Phi_{PL}B^{PL})(1-B^L)^D X_t$$
$$+\sum_{j=1}^{m} w_j\, \tilde{L}(a_j)\ f(a_j+a-a_m-\mu-(\Theta_L B^L+...+\Theta_{QL}B^{QL})\ \xi_t + \sum_{k=1}^{D}(-1)^{Lk}B^{Lk}X_t - (\Phi_L B^L+...+\Phi_{PL}B^{PL})(1-B^L)^D X_t).$$

The above equation is the system of m linear equations, which is solved by the matrix form as follows:

$$\mathbf{L}_{m\times 1} = \mathbf{1}_{m\times 1} + \mathbf{R}_{m\times m}\mathbf{L}_{m\times 1} \tag{9}$$

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where $\mathbf{L}_{m\times 1} = \begin{pmatrix} \tilde{L}(a_1) \\ \tilde{L}(a_2) \\ \vdots \\ \tilde{L}(a_m) \end{pmatrix}$, $\mathbf{1}_{m\times 1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

$$\mathbf{R} = \begin{bmatrix} F(a-a_1-\mu-(\Theta_L \mathrm{B}^L+...+\Theta_{QL}\mathrm{B}^{QL})\ \xi_t + \sum_{k=1}^{D}(-1)^k B^k X_t - (\Phi_L \mathrm{B}^L+...+\Phi_{PL}\mathrm{B}^{PL})(1\text{-}\mathrm{B}^L)^D X_t) + & \cdots & w_m f(a_m+a-a_1-\mu-(\Theta_L \mathrm{B}^L+...+\Theta_{QL}\mathrm{B}^{QL})\ \xi_t + \sum_{k=1}^{D}(-1)^k B^k X_t - (\Phi_L \mathrm{B}^L+...+\Phi_{PL}\mathrm{B}^{PL})(1\text{-}\mathrm{B}^L)^D X_t) \\ F(a-a_1-\mu-(\Theta_L \mathrm{B}^L+...+\Theta_{QL}\mathrm{B}^{QL})\ \xi_t + \sum_{k=1}^{D}(-1)^k B^k X_t - (\Phi_L \mathrm{B}^L+...+\Phi_{PL}\mathrm{B}^{PL})(1\text{-}\mathrm{B}^L)^D X_t) + & \cdots & w_m f(a_m+a-a_2-\mu-(\Theta_L \mathrm{B}^L+...+\Theta_{QL}\mathrm{B}^{QL})\ \xi_t + \sum_{k=1}^{D}(-1)^k B^k X_t - (\Phi_L \mathrm{B}^L+...+\Phi_{PL}\mathrm{B}^{PL})(1\text{-}\mathrm{B}^L)^D X_t) \\ \vdots & \cdots & \\ F(a-a_m-\mu-(\Theta_L \mathrm{B}^L+...+\Theta_{QL}\mathrm{B}^{QL})\ \xi_t + \sum_{k=1}^{D}(-1)^k B^k X_t - (\Phi_L \mathrm{B}^L+...+\Phi_{PL}\mathrm{B}^{PL})(1\text{-}\mathrm{B}^L)^D X_t) + & \cdots & w_m f(a_m+a-a_m-\mu-(\Theta_L \mathrm{B}^L+...+\Theta_{QL}\mathrm{B}^{QL})\ \xi_t + \sum_{k=1}^{D}(-1)^k B^k X_t - (\Phi_L \mathrm{B}^L+...+\Phi_{PL}\mathrm{B}^{PL})(1\text{-}\mathrm{B}^L)^D X_t) \end{bmatrix}$$

and $\mathbf{I}_m = diag(1,1,...,1)$. If $(\mathbf{I}_m - \mathbf{R}_{m\times m})^{-1}$ there exist

$$\mathbf{L}_{m\times 1} = (\mathbf{I}_m - \mathbf{R}_{m\times m})^{-1}\mathbf{1}_{m\times 1}. \tag{10}$$

Finally, the numerical integral equation of ARL of CUSUM control chart for process is given by

$$\tilde{L}(u) = 1 + \tilde{L}(a_1)\, F(a-u-\mu-(\Theta_L \mathrm{B}^L+...+\Theta_{QL}\mathrm{B}^{QL})\ \xi_t + \sum_{k=1}^{D}(-1)^k B^k X_t - (\Phi_L \mathrm{B}^L+...+\Phi_{PL}\mathrm{B}^{PL})(1\text{-}\mathrm{B}^L)^D X_t$$
$$+ \sum_{j=1}^{m} w_j\, \tilde{\mathrm{L}}(a_j)\ f(a_j+a-u-\mu-(\Theta_L \mathrm{B}^L+...+\Theta_{QL}\mathrm{B}^{QL})\ \xi_t + \sum_{k=1}^{D}(-1)^k B^k X_t - (\Phi_L \mathrm{B}^L+...+\Phi_{PL}\mathrm{B}^{PL})(1\text{-}\mathrm{B}^L)^D X_t). \tag{11}$$

where $w_j = \frac{h}{m}$ and $a_j = \frac{h}{m}\left(j - \frac{1}{2}\right); j = 1,2,...,m.$

Monte Carlo Simulation Method

Monte Carlo (MC) simulation methods is a powerful tool for approximating a distribution when deriving the exact one is difficult. MC method is some computational algorithms that use the process of repeated random sampling to make numerical estimations of unknown parameters. In this paper, we used a MC simulation to approximate the ARL of CUSUM control chart.

Firstly, we consider in the case of observations are SARIMA(P,D,Q)$_L$ process with exponential white noise with parameter β. In situation the process is in-control state, we set the exponential parameter $\beta = \beta_0 = 1$ and the process is out-of-control state, the exponential parameter $\beta = \beta_1 = \beta_0 + \delta\beta_0$ where δ is the magnitude of shift size; δ = 0.00, 0.10, 0.20, 0.30, 0.40, 0.50 and 1.00. According to a specified in-control average run length (ARL$_0$) is equal to 370, we determine the parameter of control charts based on the Monte Carlo simulations technique. The average run length was estimated by means of the number of simulation study (M) is equal to 10,000 times.

The run length is defined as

$$RL_l = \begin{cases} 1\ ;\ LCL_l \le C \le UCL_l \\ 0\ ;\ Otherwise. \end{cases} \tag{12}$$

where l = 1, 2, … , M.

Therefore, the average run length is described by

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$$ARL = \frac{\sum_{l=1}^{M} RL_l}{M} \quad (13)$$

where LCL_l is the lower control limit of control chart

UCL_l is the upper control limit of control chart

M is the number of the simulation study.

Comparison Results

In this section, we compare the numerical integral equation results with Monte Carlo simulation results in term of the absolute percentage difference and the computational time. We compute the numerical integral equation values for ARL from (11) with parameter (*a* and *h*) and compare these results with the values obtained from the Monte Carlo simulation method. The numerical integral equation solution is denoted by (NIE) and the Monte Carlo simulation solution is denoted by (MC).

The absolute percentage difference is given by

$$Diff\,(\%) \;=\; \frac{|NIE - MC|}{NIE} \times 100. \quad (14)$$

In this section, the numerical results of in-control average run length (ARL_0) and out-of-control average run length (ARL_1) for a CUSUM control chart were calculated from (11) and (13) as shown in Table 1 to Table 3. According to the numerical integral equation with the number of division points m = 800 nodes. The ARLs at different levels of parameter Φ and Θ calculated by keeping the in-control average run length (ARL_0) of approximately 370. We assume that process starts with in-control state, the value of the in-control parameter β_0 = 1. For example, in Table 1 if we fixed an ARL_0 = 370 for $SARIMA(1,1,1)_4$ process with parameter $\Phi_{1,4}$ = 0.10 and $\Theta_{1,4}$ = 0.10, then the parameter of CUSUM control chart (*a* = 2.00 and *h* = 4.585) are adjusted in such a way that the ARL_0 of approximately 370 may be maintained. In the case of SARIMA)1,1,1($_{12}$ process with parameter $\Phi_{1,12}$ = 0.10 and $\Theta_{1,12}$ = 0.20, we use the parameter of CUSUM chart are *a* = 2.50 and *h* = 3.529. Finally, in the case of SARIMA(1,1,1($_{12}$ process with parameter $\Phi_{1,12}$ = 0.10 and $\Theta_{1,12}$ = 0.10, therefore the parameter of CUSUM control chart are *a* = 3.00 and *h* = 3.028. Numerical values for ARL_0 are computed for an in-control parameter value and numerical values for ARL_1 are computed for range of out-of-control parameter values $\beta_1 = \beta_0 + \delta\beta_0$ where δ = 0.00, 0.10, 0.20, 0.30, 0.40, 0.50 and 1.00 respectively.

Table 1 Comparison of ARL computed using NIE method against MC method for $SARIMA(1,1,1)_{12}$ process with parameter $\Phi_{1,4}$ = 0.10, $\Theta_{1,4}$ = 0.10, *a* = 2.00 and *h* = 4.585.

Parameter		Method		
β		NIE	MC	Diff%
1.00	ARL	369.263	370.091	0.224
	(CPU time)	(47.121)	(196.251)	
1.10	ARL	190.483	190.825	0.180
	(CPU time)	(47.853)	(196.181)	
1.20	ARL	110.443	110.602	0.144
	(CPU time)	(47.447)	(193.453)	
1.30	ARL	70.238	70.319	0.115
	(CPU time)	(48.809)	(186.142)	

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1.40	ARL (CPU time)	48.094 (47.23)	48.140 (179.362)	0.096
1.50	ARL (CPU time)	34.948 (47.26)	34.975 (175.473)	0.077
2.00	ARL (CPU time)	12.462 (47.54)	12.466 (166.215)	0.032

The performance of the numerical integral equation is measured in terms of the absolute percentage difference and the computational time. From Table 1-3, it is found that the absolute percentage difference less than 1.0% between the NIE method and the MC method. In addition, the computational time based on the NIE method takes less than 60 minutes, while the MC method takes approximately 190 minutes.

Table 2 Comparison of ARL computed using NIE method against MC method for SARIMA$(1,1,1)_{12}$ process with parameter $\Phi_{1,12}$ = 0.10, $\Theta_{1,12}$ = 0.20, a = 2.50 and h = 3.529.

Parameter		Method		
β		NIE	MC	Diff%
1.00	ARL (CPU time)	370.00 (52.475)	370.045 (196.141)	0.012
1.10	ARL (CPU time)	205.446 (53.702)	205.812 (192.362)	0.178
1.20	ARL (CPU time)	126.014 (51.643)	126.213 (188.211)	0.158
1.30	ARL (CPU time)	83.403 (54.758)	83.519 (186.191)	0.139
1.40	ARL (CPU time)	58.637 (50.155)	58.711 (191.143)	0.126
1.50	ARL (CPU time)	43.284 (50.574)	43.332 (193.312)	0.111
2.00	ARL (CPU time)	15.301 (52.453)	15.311 (189.141)	0.065

Table 3 Comparison of ARL computed using NIE method against MC method for SARIMA$(1,1,1)_{12}$ process with parameter $\Phi_{1,12}$ = 0.10, $\Theta_{1,12}$ = 0.10, a = 3.00 and h = 3.028.

Parameter		Method		
β		NIE	MC	Diff%
1.00	ARL ARL	370.00 (53.151)	370.276 (199.241)	0.289
1.10	(CPU time) ARL	370.00 (52.924)	209.872 (198.763)	0.257
1.20	(CPU time) ARL	130.32 (53.107)	130.620 (199.243)	0.230
1.30	(CPU time) ARL	87.236 (52.147)	87.417 (198.756)	0.207

1.40	(CPU time) ARL	61.852 (54.014)	61.967 (197.513)	0.187
1.50	(CPU time) ARL	46.132 (53.983)	46.010 (199.421)	0.169
2.00	(CPU time) ARL	16.366 (54.015)	16.383 (199.137)	0.109

Conclusion

This research purposes the numerical integral equation for the average run length (ARL) by using Gauss-Legendre quadrature rule of the cumulative sum control chart for a seasonal autoregressive integrated moving average; SARIMA$(P,D,Q)_L$ process with exponential white noise. The results from purposed numerical integral equation method for finding the ARL of CUSUM control chart are compared with the Monte Carlo simulation approaches to verify that the accuracy of the numerical integral equation is good agreement with the Monte Carlo simulation approaches. Furthermore, the purposed method takes time much less than the Monte Carlo simulation method obviously is demonstrated in Table 1-3.

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