The Journal of Applied Science **Vol. 19 No. 2: 11-23 [2020]**
วารสารวิทยาศาสตร์ประยุกต์ doi: 10.14416/j.appsci.2020.02.002
ISSN 1513-7805 Printed in Thailand ISSN 2586-9663 (Online)

Research Article

On the Monoid of Generalized Cohypersubstitutions of type $\tau=(n)$

Nagornchat Chansuriya*

Faculty of Science, Energy and Environment, King Mongkut's University of Technology North Bangkok (Rayong Campus), Rayong 21120, Thailand
***E-mail**: nagornchat.c@sciee.kmutnb.ac.th
Received: 20/06/2020; Revised: 25/09/2020; Accepted: 05/11/2020

Abstract

In this research, we study some properties of projection and dual generalized cohypersubstitutions of type $\tau=(n)$ and characterize the set of all idempotent elements of monoid of generalized cohypersubstitutions of type $\tau=(n)$.

Keywords: generalized cohypersubstitutions, idempotent elements, projection and dual generalized cohypersubstitutions

Introduction

The topic on universal algebra, interested by many authors, is cohypersubstitution of type τ. It was firstly introduced by Denecke & Saengsura (2009). They constructed the monoid of cohypersubstitutions of type τ and studied some algebraic-structural properties of their monoid. After that, Boonchari & Saengsura (2016) focused on the monoid of cohypersubstitutions of type $\tau=(n)$, $Cohyp(n)$, and characterized all idempotent and regular elements of the monoid. They also characterized some Green's relations on $Cohyp(n)$. Afterwards, Lekkoksung, N., Jampachon & Lekkoksung, S. (2017) studied the semigroup properties and characterized all idempotent elements of weak projection cohypersubstitutions. For studying on generalized cohypersubstitutions of type $\tau=(n_i)_{i\in I}$, $Cohyp_G(\tau)$, Jermjitpornchai & Saengsura (2013) generalized the concept of Denecke & Saengsura (2009) and introduced the coterms, generalized superpositions, and some algebraic-structural properties. They also constructed the monoid of generalized cohypersubstitutions of type $\tau=(n_i)_{i\in I}$. For $\tau=(2)$, Saengsura & Jermjitpornchai (2013) characterized all idempotent and regular elements. Moreover, Saengsura & Butkote (2015) characterized all idempotent and regular elements of the monoid of generalized cohypersubstitutions of type $\tau=(3)$.

In semigroup theory, it is of interest to consider various type of its elements, including idempotent, regular, intra-regular, completely regular, etc. The set of all idempotent elements can be describe the special subsemigroup of the semigroup such as maximal subsemigroup, maximal unit regular subsemigroup, etc. Moreover, the idempotent elements are the important

 doi: 10.14416/j.appsci.2020.02.002

example of regular element that use to describe the structure of regular semigroup. In generalized cohypersubstitutions of type τ, the set of all idempotent elements is important to the study of algebraic-structural properties of the monoid $Cohyp_G(\tau)$ such as the charecterization of maximal idempotent submonoid, the charecterization of regular elements and the regular semigroup of generalized cohypersubstitutions of type τ. In the case of $\tau = (2)$, Chansuriya (2020) determined all maximal idempotent submonoids which are useful to determine another properties of the monoid. In this research, we focus on the monoid of generalized cohypersubstitutions of type $\tau = (n)$. Firstly, we study the properties of projection and dual generalized cohypersubstitutions. Furthermore, we characterize the set of all idempotent elements of this monoid which is a tool for determine some maximal submonoids and some structural properties of the monoid.

Materials and Methods

In this section, we collect the basic definitions and the concept of the monoid of generalized cohypersubstitutions for using in this research.

Definition 1. [Jermjitpornchai & Saengsura (2013)] Let A be a nonempty set and n be a positive integer. The n-th copower $A^{\grave{o}\, n}$ of A is the union of n disjoint copies of A; formally, we define $A^{\grave{o}\, n}$ as the Cartesian product $A^{\grave{o}\, n} := \underline{n} \times A$, where $\underline{n} := \{1, \ldots, n\}$, i.e. $A^{\grave{o}\, n} := \{(i,a) \mid i \in \underline{n}$ and a in the i-th copy of $A\}$.

A *co-operation* on A is a mapping $f^A : A \to A^{\grave{o}\, n}$ for some $n \geq 1$ where n is the arity of the co-operation f^A and f^A is n-ary co-operation defined on the set A. Every n-ary co-operation f^A on the set A can be uniquely expressed as the pair of mappings (f_1^A, f_2^A) where $f_1^A : A \to \underline{n}$ gives the labelling used by f^A in mapping elements to copie of A, and $f_2^A : A \to A$ tells us what element of A any element is mapped to, so $f^A(a) = (f_1^A(a), f_2^A(a))$. Note that $cO_A^{(n)} = \{f^A : A \to A^{\grave{o}\, n}\}$ is the set of all n-ary co-operations defined on A and $cO_A := \bigcup_{n\geq 1} cO_A^{(n)}$ is the set of all finitary co-operations defined on A. The special co-operations which are defined for each $0 \leq i \leq n-1$ by $\iota_i^{n,A} : A \to A^{\grave{o}\, n}$ with $a \mapsto (i,a)$ for all $a \in A$ are called *injection co-operation*.

Definition 2. [Jermjitpornchai & Saengsura (2013)] Let $\tau = (n_i)_{i\in I}$ and $(f_i)_{i\in I}$ be an indexed set of co-operation symbols which f_i has arity n_i for each $i \in I$. Let $\{e_j^n \mid n \geq 1, n \in \square, 0 \leq j \leq n-1\}$ be a set of symbols which disjoint from $\{f_i \mid i \in I\}$ such that e_j^n has arity n for each $0 \leq j \leq n-1$. The *coterms* of of type τ are defined as follows:

(i) For every $i \in I$ the co-operation symbol f_i is an n_i-ary coterm of type τ.

(ii) For every $n \geq 1$ and $0 \leq j \leq n-1$ the symbol e_j^n is an n-ary coterm of type τ.

(iii) If $t_0, \ldots, t_{n_i-1}$ are n-ary coterms of type τ, then for every $i \in I$, $f_i\left[t_0, \ldots, t_{n_i-1}\right]$ is an n-ary coterm of type τ, and if $t_0, \ldots, t_{n-1}$ are m-ary coterm of type τ, then $e_j^n\left[t_0, \ldots, t_{n-1}\right]$ is an m-ary coterm of type τ for every $0 \leq j \leq n-1$.

The set of all n-ary coterms of type τ denoted by $CT_\tau^{(n)}$, and the set of all coterms of type τ denoted by $CT_\tau := \bigcup_{n \geq 1} CT_\tau^{(n)}$. The following is a generalized superposition of the coterms as following.

Definition 3. [Jermjitpornchai & Saengsura (2013)] Let $\mathbb{N}^* = \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}^*$. A *generalized superposition* of the coterms $S^m : CT_\tau \times \left(CT_\tau\right)^m \to CT_\tau$ defined inductively by the fillowing steps:

(i) If $t = e_i^n$ and $0 \leq i \leq m-1$, then $S^m\left(e_i^n, t_0, \ldots, t_{m-1}\right) = t_i$, where $t_0, \ldots, t_{m-1} \in CT_\tau$.

(ii) If $t = e_i^n$ and $0 < m \leq i \leq n-1$, then $S^m\left(e_i^n, t_0, \ldots, t_{m-1}\right) = e_i^n$, where $t_0, \ldots, t_{m-1} \in CT_\tau$.

(iii) If $t = f_i\left[s_1, \ldots s_{n_i}\right]$, then
$S^m\left(t, t_1, \ldots, t_m\right) = f_i\left[S^m\left(s_1, t_1, \ldots, t_m\right), \ldots, S^m\left(s_{n_i}, t_1, \ldots, t_m\right)\right]$, where
$S^m\left(s_1, t_1, \ldots, t_m\right), \ldots, S^m\left(s_{n_i}, t_1, \ldots, t_m\right) \in CT_\tau$.

Now, the above definition can be written as the following forms:

(i) If $t = e_i^n$ and $0 \leq i \leq m-1$, then $e_i^n\left[t_0, \ldots, t_{m-1}\right] = t_i$, where $t_0, \ldots, t_{m-1} \in CT_\tau$.

(ii) If $t = e_i^n$ and $0 < m \leq i \leq n-1$, then $e_i^n\left[t_0, \ldots, t_{m-1}\right] = e_i^n$, where $t_0, \ldots, t_{m-1} \in CT_\tau$.

(iii) If $t = f_i\left[s_1, \ldots s_{n_i}\right]$, then
$\left(f_i\left[s_1, \ldots, s_{n_i}\right]\right)\left[t_1, \ldots, t_m\right] = f_i\left[s_1\left[t_1, \ldots, t_m\right], \ldots, s_{n_i}\left[t_1, \ldots, t_m\right]\right]$, where
$s_1\left[t_1, \ldots, t_m\right], \ldots, s_{n_i}\left[t_1, \ldots, t_m\right] \in CT_\tau$.

Definition 4. [Jermjitpornchai & Saengsura (2013)] A *generalized cohypersubstitution* of type τ is a mapping $\sigma : \left\{ f_i \mid i \in I \right\} \to CT_\tau$. Denoted by $Cohyp_G(\tau)$ the set of all generalized

 doi: 10.14416/j.appsci.2020.02.002

cohypersubstitutions of type τ. If $t, t_1, \ldots, t_n \in CT_\tau$ and $\sigma \in Cohyp_G(\tau)$, then $\hat{\sigma}(t[t_1, \ldots, t_n]) := \hat{\sigma}(t)[\hat{\sigma}(t_1), \ldots, \hat{\sigma}(t_n)]$.

Generalized cohypersunstitution σ can be uniquely extended to mappings $\hat{\sigma} : CT_\tau \to CT_\tau$ which are inductively defined by the following steps:

(i) $\hat{\sigma}(e_j^n) := e_j^n$ for every $n \geq 1$ and $0 \leq j \leq n-1$,

(ii) $\hat{\sigma}(f_i) := \sigma(f_i)$ for every $i \in I$,

(iii) $\hat{\sigma}(f_i[t_1, \ldots, t_{n_i}]) := \sigma(f_i)[\hat{\sigma}(t_1), \ldots, \hat{\sigma}(t_{n_i})]$ for $t_1, \ldots, t_{n_i} \in CT_\tau^{(n)}$.

Using this extension of generalized cohypersubstitution we can define a binary operation on the set $Cohyp_G(\tau)$ of all generalized cohypersunstitutions of type τ as following. Define a function $\circ_{CG} : Cohyp_G(\tau) \times Cohyp_G(\tau) \to Cohyp_G(\tau)$ on the set of all generalized cohypersubstitutions by $\sigma_1 \circ_{CG} \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ for all $\sigma_1, \sigma_2 \in Cohyp_G(\tau)$ where $\circ$ is a usual composition of mappings. Let σ_{id} be the generalized cohypersubstitution such that $\sigma_{id}(f_i) := f_i$ for all $i \in I$. Then the following proposition is obtained.

Proposition 5. [Jermjitpornchai & Saengsura (2013)] The set $Cohyp_G(\tau)$ of all generalized cohypersubstitutions of type τ is associates with a binary operation $\circ_{CG}$ and the generalized cohypersubtitution σ_{id} is an identity of $Cohyp_G(\tau)$.

So $\underline{Cohyp_G(\tau)} := (Cohyp_G(\tau), \circ_{CG}, \sigma_{id})$ is a monoid. This monoid is called the monoid of all generalized cohypersubstitutions of type τ. For more details of the monoid of all generalized cohypersubstitutions of type τ see in Jermjitpornchai & Saengsura (2013).

Results

In this section, we denote:

f := a cooperation symbol of type $\tau = (n)$,

σ_t := the generalized cohypersubstitution σ of type τ which maps f to the coterm t,

e_j^n := the injection symbol for all $0 \leq j \leq n-1$, $n \in \mathbb{N}$,

$\sigma_{e_j^n}$:= the generalized cohypersubstitution σ of type τ which maps f to the coterm e_j^n,

$E(t)$:= the set of all injection symbols which occure in the coterm t: i.e. for $\tau = (2)$, if $t = f[e_0^2, f[e_2^2, e_4^2]]$, then $E(t) = \{e_0^2, e_2^2, e_4^2\}$.

Now, we consider two types of special elements as following.

Vol. 19 No. 2: 11-23 [2020]
doi: 10.14416/j.appsci.2020.02.002

1. Projection and dual generalized cohypersubstitutions of type $\tau=(n)$

In this section, we study the set of special elements of $Cohyp_G(n)$ that are projection and dual generalized cohypersubstitutions.

Let $\tau=(n)$ and $E:=\{e_j^n \mid n, j\in \mathbb{N}\}$. A *projection generalized cohypersubstitution* is a generalized cohypersubstitution σ_t where t is the injection symbol. Denoted by $P_G^{inj}(n)$ the set of all projection generalized cohypersubstitutions of type $\tau=(n)$, i.e. $P_G^{inj}(n):=\left\{\sigma_{e_j^n} \mid e_j^n\in E\right\}$.

Lemma 1. For any $\sigma_t\in Cohyp_G(n)$ and $\sigma_{e_j^n}\in P_G^{inj}(n)$, we have

(i) $\sigma_t\circ_{CG}\sigma_{e_j^n}=\sigma_{e_j^n}$

(ii) $\sigma_{e_j^n}\circ_{CG}\sigma_t\in P_G^{inj}(n)$.

Proof. (i) Let $\sigma_{e_j^n}(f)=e_j^n$. Consider

$$\left(\sigma_t\circ_{CG}\sigma_{e_j^n}\right)(f)=\left(\hat{\sigma}_t\circ\sigma_{e_j^n}\right)(f)=\hat{\sigma}_t\left(\sigma_{e_j^n}(f)\right)=\hat{\sigma}_t\left(e_j^n\right)=e_j^n=\sigma_{e_j^n}(f).$$

Thus $\sigma_t\circ_{CG}\sigma_{e_j^n}=\sigma_{e_j^n}$.

(ii) We will consider two cases of t. If $t\in E$ and (i), then we have $\sigma_{e_j^n}\circ_{CG}\sigma_t=\sigma_t\in P_G^{inj}(n)$. Assume that $t=f[s_0,\ldots,s_{n-1}]$ and $\sigma_{e_j^n}\circ_{CG}\sigma_{s_0},\ldots,\sigma_{e_j^n}\circ_{CG}\sigma_{s_{n-1}}\in P_G^{inj}(n)$. So $\hat{\sigma}_{e_j^n}(s_0),\ldots,\hat{\sigma}_{e_j^n}(s_{n-1})\in E$. Then we get

$$\begin{aligned}\left(\sigma_{e_j^n}\circ_{CG}\sigma_t\right)(f)&=\left(\sigma_{e_j^n}\circ_{CG}\sigma_{f[s_0,\ldots,s_{n-1}]}\right)(f)\\&=\hat{\sigma}_{e_j^n}\left(f[s_0,\ldots,s_{n-1}]\right)\\&=e_j^n\left(\hat{\sigma}_{e_j^n}[s_0],\ldots,\hat{\sigma}_{e_j^n}[s_{n-1}]\right).\end{aligned}$$

If $0\le j\le n-1$, then $\left(\sigma_{e_j^n}\circ_{CG}\sigma_t\right)(f)=\hat{\sigma}_{e_j^n}(s_j)\in E$. If $j\ge n$, then $\left(\sigma_{e_j^n}\circ_{CG}\sigma_t\right)(f)=e_j^n\in E$. So $\sigma_{e_j^n}\circ_{CG}\sigma_t\in P_G^{inj}(n)$.

Corollary 2.

(i) $P_G^{inj}(n)\cup\{\sigma_{id}\}$ is a submonoid of $Cohyp_G(n)$ and $P_G^{inj}(n)$ is the smallest two-side ideal of $Cohyp_G(n)$.

 doi: 10.14416/j.appsci.2020.02.002

(ii) $P_G^{inj}(n)$ is the set of all right-zero elements of $Cohyp_G(n)$.

(iii) There are no left-zero elements in $Cohyp_G(n)$.

Proof. This proof is following from Lemma 1.

Next, we study another special kind of generalized cohypersubstitutions in $Cohyp_G(n)$. Let π be a permutation of the set $J=\{0,1,\ldots,n-1\}$. For any such permutation π, the generalized cohypersubstitution σ_π is called *dual generalized cohypersubstitutions* if $\sigma_\pi = \sigma_{f\left[e^n_{\pi(0)},e^n_{\pi(1)},\ldots,e^n_{\pi(n-1)}\right]}$. Denoted by D_G^{inj} the set of all such dual generalized cohypersubstitutions.

Lemma 3.

(i) For any permutations π and γ, we have $\sigma_\pi \circ_{CG} \sigma_\gamma = \sigma_{\pi\circ\gamma}$.

(ii) Let π be any permutation with the inverse π^{-1}. The generalized cohypersubstitutions σ_π and $\sigma_{\pi^{-1}}$ are inverse of each other.

Proof. (i) Let $\sigma_\pi, \sigma_\gamma \in D_G^{inj}$ such that $\sigma_\pi = \sigma_{f\left[e^n_{\pi(0)},\ldots,e^n_{\pi(n-1)}\right]}$ and $\sigma_\gamma = \sigma_{f\left[e^n_{\gamma(0)},\ldots,e^n_{\gamma(n-1)}\right]}$. We consider

$$\begin{aligned}(\sigma_\pi \circ_{CG} \sigma_\gamma)(f) &= \hat{\sigma}_\pi\left(f\left[e^n_{\gamma(0)},\ldots,e^n_{\gamma(n-1)}\right]\right)\\ &= (\sigma_\pi(f))\left[\hat{\sigma}_\pi\left(e^n_{\gamma(0)}\right),\ldots,\hat{\sigma}_\pi\left(e^n_{\gamma(n-1)}\right)\right]\\ &= \left(f\left[e^n_{\pi(0)},\ldots,e^n_{\pi(n-1)}\right]\right)\left[e^n_{\gamma(0)},\ldots,e^n_{\gamma(n-1)}\right]\\ &= f\left[e^n_{\pi(0)}\left[e^n_{\gamma(0)},\ldots,e^n_{\gamma(n-1)}\right],\ldots,e^n_{\pi(n-1)}\left[e^n_{\gamma(0)},\ldots,e^n_{\gamma(n-1)}\right]\right]\\ &= f\left[e^n_{\pi(\gamma(0))},\ldots,e^n_{\pi(\gamma(n-1))}\right]\\ &= \sigma_{\pi\circ\gamma}(f).\end{aligned}$$

Thus $\sigma_\pi \circ_{CG} \sigma_\gamma = \sigma_{\pi\circ\gamma}$.

(ii) These follows from (i).

Lemma 4. Let σ and δ be in $Cohyp_G(n)$. If $\sigma \circ_{CG} \delta \in D_G^{inj}$, then σ and δ are all in D_G^{inj}.

Proof. Let $\sigma(f) = f[u_0,\ldots,u_{n-1}]$ and $\delta(f) = f[v_0,\ldots,v_{n-1}]$.

Consider

$$\begin{aligned}(\sigma \circ_{CG} \delta)(f) &= \hat{\sigma}(f[v_0,\ldots,v_{n-1}]) \\ &= (\sigma(f))[\hat{\sigma}(v_0),\ldots,\hat{\sigma}(v_{n-1})] \\ &= (f[u_0,\ldots,u_{n-1}])[\hat{\sigma}(v_0),\ldots,\hat{\sigma}(v_{n-1})] \\ &= f[u_0[\hat{\sigma}(v_0),\ldots,\hat{\sigma}(v_{n-1})],\ldots,u_{n-1}[\hat{\sigma}(v_0),\ldots,\hat{\sigma}(v_{n-1})]].\end{aligned}$$

Since $\sigma \circ_{CG} \delta \in D_G^{inj}$, then there exist a permutation π such that $(\sigma \circ_{CG} \delta)(f) = f[e^n_{\pi(0)},\ldots,e^n_{\pi(n-1)}]$. So we have

$$\begin{aligned}f[e^n_{\pi(0)},\ldots,e^n_{\pi(n-1)}] &= (\sigma \circ_{CG} \delta)(f) \\ &= f[u_0[\hat{\sigma}(v_0),\ldots,\hat{\sigma}(v_{n-1})],\ldots,u_{n-1}[\hat{\sigma}(v_0),\ldots,\hat{\sigma}(v_{n-1})]].\end{aligned}$$

Since π is a permutation, then this force all the u_i's and v_i's to be the injection symbols. Therefore, σ and δ are in D_G^{inj}.

Lemma 5. $\underline{D_G^{inj}}$ is a submonoid of $\underline{Cohyp_G(n)}$ and it is a group. The elements of $Cohyp_G(n)$ doesn't have inverse in $Cohyp_G(n)$, so $\underline{D_G^{inj}}$ is a maximal subgroup of $\underline{Cohyp_G(n)}$.

Proof. By Lemma 4., we obtain that $\underline{D_G^{inj}}$ is a submonoid of $\underline{Cohyp_G(n)}$ and, by Lemma 3.(ii), we have $\underline{D_G^{inj}}$ is a group. Let D be a subgroup of $\underline{Cohyp_G(n)}$ such that $\underline{D_G^{inj}} \subseteq D \subset \underline{Cohyp_G(n)}$. Let $\sigma_t \in D$ where $t = f[u_0,...,u_{n-1}]; u_i \in CT_{(n)}, i \in \{0,...,n-1\}$. Then there exist $\sigma_{t'} \in D$ where $t' = f[u'_0,...,u'_{n-1}]; u'_i \in CT_{(n)}, i \in \{0,...,n-1\}$ such that $\sigma_t \circ_{CG} \sigma_{t'} = \sigma_{id}$. Assume that $\sigma_t \notin \underline{D_G^{inj}}$. We consider

$$\begin{aligned}(\sigma_t \circ_{CG} \sigma_{t'})(f) &= \hat{\sigma}_t(f[u'_0,...,u'_{n-1}]) \\ &= (\sigma_t(f))[\hat{\sigma}_t(u'_0),...,\hat{\sigma}_t(u'_{n-1})] \\ &= (f[u_0,...,u_{n-1}])[\hat{\sigma}_t(u'_0),...,\hat{\sigma}_t(u'_{n-1})] \\ &= f[u_0[\hat{\sigma}_t(u'_0),...,\hat{\sigma}_t(u'_{n-1})],...,u_{n-1}[\hat{\sigma}_t(u'_0),...,\hat{\sigma}_t(u'_{n-1})]].\end{aligned}$$

Since $\sigma_t \notin \underline{D_G^{inj}}$, this force that $u_j[\hat{\sigma}_t(u'_0),...,\hat{\sigma}_t(u'_{n-1})] \neq e^n_j;\ j \in \{0,...,n-1\}$. So $\sigma_t \circ_{CG} \sigma_{t'} \neq \sigma_{id}$ which is a contradiction. Thus $\sigma_t \in \underline{D_G^{inj}}$. Hence $D = \underline{D_G^{inj}}$. Therefore, $\underline{D_G^{inj}}$ is a maximal subgroup.

Let F^{inj} be the set of generalized cohypersubstitutions of the form $\sigma_{f[e^n_i,\ldots,e^n_i]}$ for $0 \le i \le n-1$. Then we have the following lemma.

 doi: 10.14416/j.appsci.2020.02.002

Lemma 6. Let $M^{inj} = P_G^{inj}(n) \cup D_G^{inj} \cup F^{inj}$. Then M^{inj} is a submonoid of $Cohyp_G(n)$.
Proof. The proof of this lemma is straightforward.

2. Idempotent in monoid of generalized cohypersubstitutions of type $\tau = (n)$

In this section, we characterize the idempotent generalized cohypersubstitutions of type $\tau = (n)$. We firstly recall the definition of an idempotent element of semigroup for using in this section. Let S be a semigroup. An element $a \in S$ is called *idempotent* if $aa = a$. Denoted by $\mathrm{E}(S)$ the set of all idempotent elements of S.

Lemma 7. Let $t, u_0, \ldots, u_{n-1}$ be in $CT_{(n)}$ and $J = \{0,1,\ldots,n-1\}$.

(i) If $E(t) = \{e_j^n \mid \forall j \in J\}$ and $u_j = e_j^n$ for all $j \in J$, then $t[u_0,\ldots,u_{n-1}] = t$.

(ii) If $E(t) \cap \{e_0^n,\ldots,e_{n-1}^n\} = \varnothing$, then $t[u_0,\ldots,u_{n-1}] = t$.

Proof. (i) We proof that by considered the cases of the coterm t.
If $t = e_j^n$ where $0 \le j \le n-1$, then $e_j^n[u_0,\ldots,u_{n-1}] = u_j = e_j^n$.
If $t = e_j^n$ where $j \ge n$, then $e_j^n[u_0,\ldots,u_{n-1}] = e_j^n$.
If $t = f[t_0,\ldots,t_{n-1}]$ and assume that $t_i[u_0,\ldots,u_{n-1}] = t_i$ where $i = 0,1,\ldots$, then

$$\begin{aligned} t[u_0,\ldots,u_{n-1}] &= (f[t_0,\ldots,t_{n-1}])[u_0,\ldots,u_{n-1}] \\ &= f[t_0[u_0,\ldots,u_{n-1}],\ldots,t_{n-1}[u_0,\ldots,u_{n-1}]] \\ &= f[t_0,\ldots,t_{n-1}] \\ &= t. \end{aligned}$$

The proof of (ii) is similarly to (i).

Theorem 8. The generalized cohypersubstitutions σ_t is an idempotent if and only if $\sigma_t(t) = t$.

Proof. Let $t \in CT_{(n)}$. Assume that σ_t is an idempotent. Then

$$\hat{\sigma}_t(t) = \hat{\sigma}_t(\sigma_t(f)) = (\sigma_t \circ_{CG} \sigma_t)(f) = \sigma_t(f) = t .$$

Conversely, assume that $\hat{\sigma}_t(t) = t$. Then

$$(\sigma_t \circ_{CG} \sigma_t)(f) = \hat{\sigma}_t(\sigma_t(t)) = \hat{\sigma}_t(t) = t = \sigma_t(f) .$$

Corollary 9.

(i) $\sigma_{e_i^n}; i = 0,1,\ldots,n-1$ and σ_{id} are idempotent.

(ii) If $\sigma_t \in Cohyp_G(n)$ and $E(t) \cap \{e_0^n, e_1^n, \ldots, e_{n-1}^n\} = \varnothing$, then σ_t is an idempotent.

 doi: 10.14416/j.appsci.2020.02.002

Proof. (i) Since $\hat{\sigma}_t\left(e_i^n\right)=e_i^n$ for all $0\leq i\leq n-1$ and $t\in CT_{(n)}$, then we have $\hat{\sigma}_{e_i^n}\left(e_i^n\right)=e_i^n$ for all $0\leq i\leq n-1$. By Theorem 8, we obtain that e_i^n is an idempotent for all $0\leq i\leq n-1$. Since σ_{id} is an identity of $Cohyp_G(n)$, then it is also an idempotent of $Cohyp_G(n)$.

(ii) Let $\sigma_t\in Cohyp_G(n)$ such that $t=[t_0,\ldots,t_{n-1}]$. Assume that $E(t)\cap\left\{e_0^n,\ldots,e_{n-1}^n\right\}=\varnothing$. Then we obtain that

$$\begin{aligned}(\sigma_t\circ_{CG}\sigma_t)(f)&=\hat{\sigma}_t\left(f[t_0,\ldots,t_{n-1}]\right)\\&=\sigma_t(f)\left[\hat{\sigma}_t(t_0),\ldots,\hat{\sigma}_t(t_{n-1})\right]\\&=\left(f[t_0,\ldots,t_{n-1}]\right)\left[\hat{\sigma}_t(t_0),\ldots,\hat{\sigma}_t(t_{n-1})\right]\\&=f[t_0,\ldots,t_{n-1}]\qquad\text{(Lemma 7.(ii))}\\&=\sigma_t(f).\end{aligned}$$

Therefore, σ_t is an idempotent.

Theorem 10. If $t=f[t_0,\ldots,t_{n-1}]$ and $E(t)\cap\left\{e_0^n,e_1^n,\ldots,e_{n-1}^n\right\}=\left\{e_j^n\right\}$ for some $j\in\{0,1,\ldots,n-1\}$, then σ_t is an idempotent if and only if $t_j=e_j^n$.

Proof. Assume that σ_t is an idempotent. Then

$$\begin{aligned}f\left[t_0,\ldots,t_j,\ldots,t_{n-1}\right]&=\sigma_t(f)=(\sigma_t\circ_{CG}\sigma_t)(f)=\hat{\sigma}_t\left(\sigma_t(f)\right)\\&=\sigma_t(f)\left[\hat{\sigma}_t(t_0),\ldots,\hat{\sigma}_t(t_j),\ldots,\hat{\sigma}_t(t_{n-1})\right]\\&=\left(f\left[t_0,\ldots,t_j,\ldots,t_{n-1}\right]\right)\left[\hat{\sigma}_t(t_0),\ldots,\hat{\sigma}_t(t_j),\ldots,\hat{\sigma}_t(t_{n-1})\right]\\&=f\Big[t_0\left[\hat{\sigma}_t(t_0),\ldots,\hat{\sigma}_t(t_j),\ldots,\hat{\sigma}_t(t_{n-1})\right],\ldots,\\&\qquad t_j\left[\hat{\sigma}_t(t_0),\ldots,\hat{\sigma}_t(t_j),\ldots,\hat{\sigma}_t(t_{n-1})\right],\ldots,\\&\qquad t_{n-1}\left[\hat{\sigma}_t(t_0),\ldots,\hat{\sigma}_t(t_j),\ldots,\hat{\sigma}_t(t_{n-1})\right]\Big].\end{aligned}$$

Then $t_j=t_j\left[\hat{\sigma}_t(t_0),\ldots,\hat{\sigma}_t(t_j),\ldots,\hat{\sigma}_t(t_{n-1})\right]$.

Suppose that $t_j\neq e_j^n$. Then we have the following two cases.

doi: 10.14416/j.appsci.2020.02.002

Case 1. If $t_i = e_l^n$ for $l \geq n$, then $\hat{\sigma}_t\left(t_j\right) = t_j$. Since $E\left(t\right) \cap \left\{e_0^n, \ldots, e_{n-1}^n\right\} = \left\{e_j^n\right\}$, then $e_j^n \in E\left(t_j\right)$ for some $i \in \{0, \ldots, n-1\}$ and $i \neq j$. So $t_j\left[\hat{\sigma}_t\left(t_0\right), \ldots, \hat{\sigma}_t\left(t_j\right), \ldots, \hat{\sigma}_t\left(t_{n-1}\right)\right] \neq t_j$, which is a contradiction.

Case 2. If the number of cooperation symbols f which occur in a coterm t_j is greater than or equal to 1, then $\hat{\sigma}_t\left(t_j\right) \neq e_j^n$. This forces that $t_j\left[\hat{\sigma}_t\left(t_0\right), \ldots, \hat{\sigma}_t\left(t_j\right), \ldots, \hat{\sigma}_t\left(t_{n-1}\right)\right] \neq t_j$, this is a contradiction. Therefore $t_j = e_j^n$.

Conversely, let $t_j = e_j^n$. Then

$$\begin{aligned}
\hat{\sigma}_t\left(t\right) &= \hat{\sigma}_t\left(f\left[t_0, \ldots, t_j, \ldots, t_{n-1}\right]\right) \\
&= \left(\sigma_t\left(f\right)\right)\left[\hat{\sigma}_t\left(t_0\right), \ldots, \hat{\sigma}_t\left(t_j\right), \ldots, \hat{\sigma}_t\left(t_{n-1}\right)\right] \\
&= \left(f\left[t_0, \ldots, e_j^n, \ldots, t_{n-1}\right]\right)\left[\hat{\sigma}_t\left(t_0\right), \ldots, e_j^n, \ldots, \hat{\sigma}_t\left(t_{n-1}\right)\right] \\
&= f\left[t_0, \ldots, e_j^n, \ldots, t_{n-1}\right] \qquad \text{(Lemma 7.(i))} \\
&= t.
\end{aligned}$$

Lemma 11. Let $t \in CT_{(n)}$. If $u_0, \ldots, u_{n-1} \in CT_{(n)}$ such that $u_j \neq e_j^n$ for some $j \in J$, $J \subseteq \{0, \ldots, n-1\}$ and $E\left(t\right) \cap \left\{e_0^n, \ldots, e_{n-1}^n\right\} = \left\{e_j^n \,\middle|\, \forall j \in J, \left|J\right| \geq 2\right\}$, then $t\left[u_0, \ldots, u_{n-1}\right] \neq t$.

Proof. Let $t = f\left[t_0, \ldots, t_{n-1}\right]$. If $t_0, \ldots, t_{n-1} \in \left\{e_0^n, \ldots, e_{n-1}^n\right\} \cup \left\{e_j^n \mid j \geq n\right\}$, then

$$\begin{aligned}
t\left[u_0, \ldots, u_{n-1}\right] &= \left(f\left[t_0, \ldots, t_{n-1}\right]\right)\left[u_0, \ldots, u_{n-1}\right] \\
&= f\left[t_0\left[u_0, \ldots, u_{n-1}\right], \ldots, t_{n-1}\left[u_0, \ldots, u_{n-1}\right]\right].
\end{aligned}$$

Since $t_0, \ldots, t_{n-1} \in \left\{e_0^n, \ldots, e_{n-1}^n\right\} \cup \left\{e_j^n \mid j \geq n\right\}$, then we get $t_j\left[u_0, \ldots, u_{n-1}\right] = u_j$ for some $j \in \{0, \ldots, n-1\}$. So $t = f\left[t_0, \ldots, t_{n-1}\right] \neq t\left[u_0, \ldots, u_{n-1}\right]$.

Assume that $t_k\left[u_0, \ldots, u_{n-1}\right] \neq t_k$ for some $k \in \{0, \ldots, n-1\}$ where $e_j^n \in E\left(t_k\right)$. Then

$$\begin{aligned}
t\left[u_0, \ldots, u_{n-1}\right] &= \left(f\left[t_0, \ldots, t_{n-1}\right]\right)\left[u_0, \ldots, u_{n-1}\right] \\
&= f\left[t_0\left[u_0, \ldots, u_{n-1}\right], \ldots, t_{n-1}\left[u_0, \ldots, u_{n-1}\right]\right] \\
&\neq f\left[t_0, \ldots, t_{n-1}\right] = t.
\end{aligned}$$

 doi: 10.14416/j.appsci.2020.02.002

Theorem 12. Let $J \subseteq \{0,1,\ldots,n-1\}$ and $t \in CT_{(n)}$ such that $E(t)\cap\{e_0^n,\ldots,e_{n-1}^n\} = \{e_j^n \mid j \in J, |J| \geq 2\}$. Then $\sigma_t \in Cohyp_G(n)$ is an idempotent if and only if $t_j = e_j^n$ for all $j \in J$.

Proof. Let $t = f[t_0,\ldots,t_{n-1}]$ and assume that σ_t is an idempotent. Then, by the same proof of Theorem 10, we obtain that

$$\begin{aligned} f[t_0,\ldots,t_j,\ldots,t_{n-1}] = f\Big[&t_0\big[\hat{\sigma}_t(t_0),\ldots,\hat{\sigma}_t(t_j),\ldots,\hat{\sigma}_t(t_{n-1})\big],\ldots,\\ &t_j\big[\hat{\sigma}_t(t_0),\ldots,\hat{\sigma}_t(t_j),\ldots,\hat{\sigma}_t(t_{n-1})\big],\ldots,\\ &t_{n-1}\big[\hat{\sigma}_t(t_0),\ldots,\hat{\sigma}_t(t_j),\ldots,\hat{\sigma}_t(t_{n-1})\big]\Big]. \end{aligned}$$

Suppose that $t_j \neq e_j^n$ for some $j \in J$. Then $\hat{\sigma}_t(t_j) \neq e_j^n$. Since $e_j^n \in E(t)$, Lemma 11 gives us that $t_l\big[\hat{\sigma}_t(t_0),\ldots,\hat{\sigma}_t(t_j),\ldots,\hat{\sigma}_t(t_{n-1})\big] \neq t_l$ for some $l \in \{0,\ldots,n-1\}$ and $e_l^n \in E(t_l)$. This implies that

$$\begin{aligned} f[t_0,\ldots,t_j,\ldots,t_{n-1}] \neq f\Big[&t_0\big[\hat{\sigma}_t(t_0),\ldots,\hat{\sigma}_t(t_j),\ldots,\hat{\sigma}_t(t_{n-1})\big],\ldots,\\ &t_j\big[\hat{\sigma}_t(t_0),\ldots,\hat{\sigma}_t(t_j),\ldots,\hat{\sigma}_t(t_{n-1})\big],\ldots,\\ &t_{n-1}\big[\hat{\sigma}_t(t_0),\ldots,\hat{\sigma}_t(t_j),\ldots,\hat{\sigma}_t(t_{n-1})\big]\Big], \end{aligned}$$

which is a contradictions. Therefore, $t_j = e_j^n$ for all $j \in J$.

Conversely, we assume that $t_j = e_j^n$ for all $j \in J$. Then $\hat{\sigma}_t(t_j) = e_j^n$. Since $E(t)\cap\{e_0^n,\ldots,e_{n-1}^n\} = \{e_j^n \mid j \in J \in \{0,1,\ldots,n-1\}, |J| \geq 2\}$, then by using Lemma 7, we obtain that

$$\begin{aligned} \hat{\sigma}_t(t) &= \hat{\sigma}_t(f[t_0,\ldots,t_{n-1}])\\ &= (f[t_0,\ldots,t_{n-1}])\big[\hat{\sigma}_t(t_0),\ldots,\hat{\sigma}_t(t_{n-1})\big]\\ &= f[t_0,\ldots,t_{n-1}] = t. \end{aligned}$$

Now, we have three disjoint sets of idempotent elements as follow:

$$E_0 := \{\sigma_{e_j^n} \mid j \in \{0,1,\ldots,n-1\}\} \cup \{\sigma_{id}\},$$

$E_1 := \{\sigma_t \mid t = f[t_0,\ldots,t_{n-1}]$ where $E(t)\cap\{e_0^n,e_1^n,\ldots,e_{n-1}^n\} = \{e_j^n \mid j \in J \subseteq \{0,1,\ldots,n-1\}$ and $t_j = e_j^n\}\}$, and

$$E_2 := \left\{ \sigma_t \,\middle|\, E(t) \cap \left\{ e_0^n, e_1^n, \ldots, e_{n-1}^n \right\} = \varnothing \right\}.$$

Then we have the following theorem.

Theorem 13. $\mathrm{E}\left(Cohyp_G(n)\right) := E_0 \cup E_1 \cup E_2$ is the set of all idempotent elements of $Cohyp_G(n)$.

Proof. The proof is directly from Corollary 9., Theorem 10., and Theorem 12.

Conclusion

This research focuses on the special elements of the monoid of generalized cohypersubstitutions of type $\tau = (n)$. Firstly, we give some algebraic-structural properties of projection and dual generalized cohypersubstitutions. After that, we characterize the idempotent elements of $Cohyp_G(n)$ and prove that $\mathrm{E}\left(Cohyp_G(n)\right) := E_0 \cup E_1 \cup E_2$ is the set of all idempotent elements of $Cohyp_G(n)$.

Acknowledgement(s)

The author would also like to thanks the referees for their careful reading of the paper and their useful comments.

References

Boonchari, D. & Saengsura, K. (2016). Monoid of cohypersubstitutions of type $\tau = (n)$. *Thai Journal of Mathematics*, 14(1), 191-201.

Chansuriya, N. (2020). All maximal idempotent sbmonoids of generalized cohypersubstitutions of type $\tau = (2)$. *preprint*.

Denecke, K. & Saengsura, K. (2009). Separation of clones of cooperations by cohyperidentities. *Discrete Mathematics*, 309, 772-783. Doi: 10.1016/j.disc.2008.01.043.

Denecke, K. & Wishmath, S.L. (2009). *Universal Algebra and Coalgebra* (1st Ed.). Singapore: World Scientific Publishing.

Jermjitpornchai, S. & Saengsura, N. (2013). Generalized cohypersubstitutions of type $\tau = (n_i)_{i \in I}$. *International Journal of Pure and Applied Mathematics*, 86(4), 747-755. Doi: http://dx.doi.org/10.12732/ijpam.v86i4.12.

Lekkoksung, N., Jampachon, P. & Lekkoksung, S. (2017). Idempotent of Weak Projection Cohypersubstitutions. *Thai Journal of Mathematics*, Special Issue: AMM2017, 59-74.

Puninagool, W. & Leeratanavalee, S. (2010). The monoid of generalized hypersubstitutions of type $\tau = (n)$. *Discuss. Math. Gen. Alg. And Appl.*, 30, 173-191. doi: 10.7151/dmgaa.1168.

Saengsura, N. & Jermjitpornchai, S. (2013). Idempotent and regular generalized cohypersubstitions of type $\tau = (2)$. *International Journal of Pure and Applied Mathematics*, 86(4), 757-766. doi: http://dx.doi.org/10.12732/ijpam.v86i4.13.

Saengsura, K. & Butkote, R. (2015). Idempotent and regular generalized cohypersubstitions of type $\tau=(3)$. *Asian-European Journal of Mathematics* 8(4), 1550081-1-1550081-11. doi: 10.1142/S1793557115500813.