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Research Article

Bayesian parameters estimation of the negative binomial - two parameter weighted exponential distribution with application in biological data

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Abstract

The purpose of this paper is to report the results of fitting three biologically significant statistical models, Poisson, negative binomial (NB) and negative binomial-two parameter weighted exponential (NB-TWE) distributions, to biological data. The results of fitting the different distributions are compared by using the K-S Test and Cramer-von Mises Test. We found that the data follows the highest value base on a NB-TWE distribution, which implies that the NB-TWE distribution is the best fit for these data sets. Therefore, the NB-TWE distribution is the best choice for biological data that has an overdispersion problem, and are heavy-tailed with positive skewness.

Keywords: count data, negative binomial distribution, mixed distribution, overdispersion, corn borer larvae data, pyrausta nubilalis

Introduction

Biology is the natural science that studies life and living organisms. Often, biological data come in the form of counts from many fields such as bacteriology (Neyman, 1939), zoology (Fisher, 1941), entomology (Taylor, 1961) and infectious diseases (Lloyed-Smith, 2007). Regression analysis can be applied for these datas, but sometimes it may not be the most appropriate for count data. A Poisson distribution is usually applied for count data modeling, and is a good model if variance and mean are equal. Many count data, especially biological data, often exhibit overdispersion, where the variance is larger than the mean. Hence a Poisson distribution is not an appropriate model. In the literature, the negative binomial (NB) distribution, which is a mixture of the Poisson and Gamma distributions, is employed as a functional form (Greenwood & Yule, 1920). It will relax the overdispersion restriction of the Poisson distribution and lead to one of the popular alternative distributions for overdispersed count data (Rainer, 2000).

In 1953, Bliss fitted the NB distribution to biological data and found that the NB distribution was the appropriate probability distribution for fitting the biological data when the variance is significantly larger than the mean, which suggests that the NB distribution is the most widely adaptable and generally useful of those that have been proposed (Bliss & Fisher, 1953). White and Bennetts modeled bird counts by using a negative binomial distribution in 1996 (White & Bennetts, 1996), whereas Alexander et al. used the negative binomial distribution with a spatial model of parasitism in 2000 (Alexander et al., 2000).

Since the mid 1980s, mixture of distributions have been widely used for modeling observed situations whose a variety of characteristics. Several studies have shown that mixed Poisson and mixed NB distributions provide a better fit compared to the Poisson and NB distributions for count data modelling. Some examples of these distributions include the Poisson-inverse Gaussian (Klugman et al., 2008), beta-negative binomial (Wang, 2011), negative binomial-beta exponential (Pudprommarat et al., 2012) and negative binomial-generalized exponential distribution (Aryuyuen & Bodhisuwan, 2013). The results of related works show the mixed Poisson and mixed NB distribution can be competitive to the Poisson and NB distributions.

In 2017, Prasongporn and Bodhisuwan (2017) introduced a new distribution: the negative binomial-two parameter weighted exponential (NB-TWE) distribution. This new distribution is heavy-tailed with positive skewness. The NB-TWE provided a better fit than the Poisson, and negative binomial distribution for overdispersion data model.

In this paper, Bayesian parameters estimation of NB-TWE is employed, and applied to biological data. The rest of paper is organized as follows. In methods, the NB-TWE and its special cases are introduced. In addition, the Bayesian approach for NB-TWE and the application with real data set are given in results and discussion and some conclusion remarks are presented in the last section.

Methods

Negative Binomial–Two Parameter Weighted Exponential (NB-TWE) Distribution

First, we present the NB distribution and some of its properties.

Let X be distributed as negative binomial with parameters n and p; its probability mass function (pmf) is

$$f(x)=\binom{r+x-1}{x} p^r (1-p)^x, x=0,1,2,..., \text{ where } r > 0 \text{ and } 0 < p < 1. \quad (1)$$

The first two moments about zero and the factorial moments of the NB distribution, respectively, are given by

$$\begin{aligned} E(X) &= \frac{r(1-p)}{p}, \\ E(X^2) &= \frac{r(1-p)\left[1+r(1-p)\right]}{p^2}, \\ \mu_{[k]}(X) &= E\left[X(X-1)\cdots(X-k+1)\right] \\ &= \frac{\Gamma(r+k)}{\Gamma(r)}\frac{(1-p)^k}{p^k}, \; k=1,2,..., \end{aligned}$$

where $\Gamma(t)=\int_0^\infty x^{t-1}\exp(-x)\,dx$ is the so-called gamma function.

Next, we introduce the NB-TWE distribution, which is gained by mixing the NB distribution with the TWE (two parameter weighted exponential) distribution. The NB and TWE distributions can be used as alternative distributions to model lifetime data. However, a limitation of these distributions occurs when the sample skewness measure for data lies outside its skewness function range. The NB-TWE distribution has a wider range of skewness than the NB and TWE distributions. The flexibility of the NB-TWE distribution and increased range of skewness are able to fit and capture features in real data sets much better than Poisson, NB and TWE distributions.

Definition 1. The pmf of the NB-TWE distribution is

$$h(x)=\binom{r+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^{j}\times\left[\frac{(1+\alpha)(1+\beta)(1+\alpha+\beta)\left(2\left(1+\frac{r+j}{\lambda}\right)+\alpha+\beta\right)}{\left(1+\frac{r+j}{\lambda}\right)\left(1+\alpha+\frac{r+j}{\lambda}\right)\left(1+\beta+\frac{r+j}{\lambda}\right)\left(1+\alpha+\beta+\frac{r+j}{\lambda}\right)(2+\alpha+\beta)}\right] \quad (2)$$

where x = 0,1,2,... and parameters λ, r, α and β > 0. (see more detail of proof in (Prasongporn, 2017))

The behavior of the NB-TWE distribution for different parameter values and some of its properties and their graphs are displayed in the following corollaries. Moreover, the shapes of the NB-TWE distribution for different parameter values are illustrated in Figure 1. It confirms that the NB-TWE distribution is positively skewed and heavy-tailed.

Next, we will present some related distributions and moments of the NB-TWE distribution.

Some related distributions

The expansion of the NB-TWE distribution is introduced in this section. We present special cases of the NB-TWE distribution. (see more detail of proofs in (Prasongporn, 2017))

Corollary 1. If $\lambda = 1$, and $\alpha, \beta \to \infty$, the NB-TWE distribution then reduces to the negative binomial-exponential distribution (NB-E) with the pmf

$$h(x)=\binom{r+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^{j}\frac{1}{1+r+j},\quad x=0,1,2,... \quad (3)$$

Corollary 2. If $\lambda = 1$, and $\alpha, \beta \to 0$, the NB-TWE distribution then reduces to the negative binomial-gamma distribution (NB-gamma) with the pmf

$$h(x)=\binom{r+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^{j}\frac{1}{(1+r+j)^{3}},\quad x=0,1,2,... \quad (4)$$

Corollary 3. If $\lambda=1$,and $\lim_{\alpha\to\infty} h(x)$,then the NB-TWE distribution reduces to the negative binomial-weighted exponential distribution (NB–WE) with the pmf

Vol. 18 No. 1: 39-48 [2019]
doi: 10.14416/j.appsci.2019.03.001

$$h(x)=\binom{r+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^{j}\left(1+\frac{r+j}{1+\beta}\right)^{-1}\frac{1}{1+r+j},\quad x=0,1,2,\ldots \tag{5}$$

Corollary 4. If λ=1,and limβ→∞ h(x) , then the NB-TWE distribution reduces to the negative binomial-weighted exponential distribution (NB-WE) with the pmf

$$h(x)=\binom{r+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^{j}\left(1+\frac{r+j}{1+\alpha}\right)^{-1}\frac{1}{1+r+j},\quad x=0,1,2,\ldots \tag{6}$$

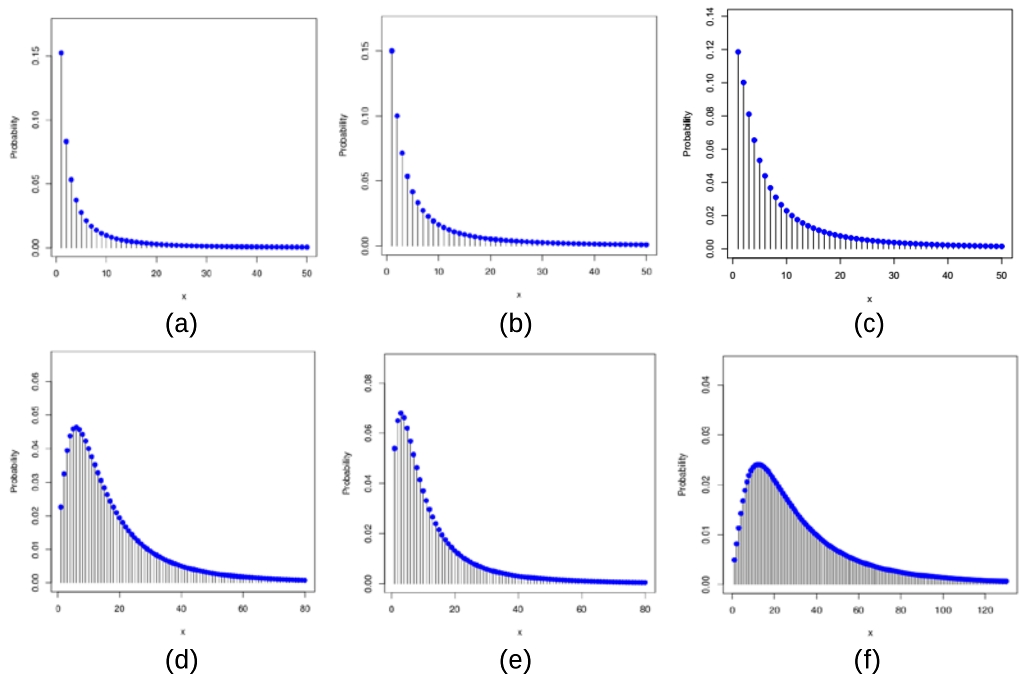

Figure 1. Plots of the NB-TWE distribution with some specified parameter values

parameter (a) is $r=0.5$, $\lambda=1$, $\alpha=1$, $\beta=1$
parameter (b) is $r=1$, $\lambda=1$, $\alpha=1$, $\beta=1$
parameter (c) is $r=5$, $\lambda=2$, $\alpha=2$, $\beta=2$
parameter (d) is $r=10$, $\lambda=2$, $\alpha=2$, $\beta=2$
parameter (e) is $r=25$, $\lambda=3$, $\alpha=2$, $\beta=2$
parameter (f) is $r=50$, $\lambda=3$, $\alpha=2$, $\beta=2$

The factorial moment

The factorial moment of order k of X If X ~ NB-TWE (r, λ, α , β). is given by

Vol. 18 No. 1: 39-48 [2019]
doi: 10.14416/j.appsci.2019.03.001

$$\mu_{[k]}(X)=\frac{\Gamma(r+k)}{\Gamma(r)}\sum_{j=0}^{k}\binom{k}{j}(-1)^{j}\times\left[\frac{(1+\alpha)(1+\beta)(1+\alpha+\beta)\left(2\left(1+\frac{r+j}{\lambda}\right)+\alpha+\beta\right)}{\left(1+\frac{r+j}{\lambda}\right)\left(1+\alpha+\frac{r+j}{\lambda}\right)\left(1+\beta+\frac{r+j}{\lambda}\right)\left(1+\alpha+\beta+\frac{r+j}{\lambda}\right)(2+\alpha+\beta)}\right]$$

where r, λ, α and β >0. (7)

Next, Bayseian parameters distribution of the NB-TWE will be presented, including prior distribution, joint density and the posterior distribution of r,λ,α,β.

Results and discussion

Bayseian Parameters Estimation Distribution

The parameters of NB-TWE distribution can also be estimated using Bayesian approach, which considers parameters as random variables that are characterized by a prior distribution. Due to parameters in NB-TWE distribution over than zero, the gamma distribution is an appropriate to the parameter space, so we select the gamma distribution as a prior distribution r,λ,α,β as shown

r ~ Gamma (a_r,b_r), which pdf $\pi(r) = \dfrac{b_r^{a_r}}{\Gamma(a_r)} r^{a_r-1}e^{-b_r r}, a_r, b_r > 0,$

λ ~ Gamma (a_λ,b_λ), which pdf $\pi(\lambda) = \dfrac{b_\lambda^{a_\lambda}}{\Gamma(a_\lambda)} \lambda^{a_\lambda-1}e^{-b_\lambda \lambda}, a_\lambda, b_\lambda > 0,$

α ~ Gamma (a_α,b_α), which pdf $\pi(\alpha) = \dfrac{b_\alpha^{a_\alpha}}{\Gamma(a_\alpha)} \alpha^{a_\alpha-1}e^{-b_\alpha \alpha}, a_\alpha, b_\alpha > 0,$

β ~ Gamma (a_β,b_β), which pdf $\pi(\beta) = \dfrac{b_\beta^{a_\beta}}{\Gamma(a_\beta)} \beta^{a_\beta-1}e^{-b_\beta \beta}, a_\beta, b_\beta > 0$

According to the likelihood function parameter of NB-TWE distribution as

$$\mathscr{L}(r,\lambda,\alpha,\beta)=\prod_{i=1}^{n}\binom{r+x_i-1}{x_i}\sum_{j=0}^{k}\binom{x_i}{j}(-1)^{j}\times\left[\frac{(1+\alpha)(1+\beta)(1+\alpha+\beta)\left(2\left(1+\frac{r+j}{\lambda}\right)+\alpha+\beta\right)}{\left(1+\frac{r+j}{\lambda}\right)\left(1+\alpha+\frac{r+j}{\lambda}\right)\left(1+\beta+\frac{r+j}{\lambda}\right)\left(1+\alpha+\beta+\frac{r+j}{\lambda}\right)(2+\alpha+\beta)}\right]$$

The joint density of the NB-TWE was expressed as follows

$$f(y_1,y_2,\ldots,y_n,r,\lambda,\alpha,\beta)=\prod_{i=1}^{n}\binom{r+x_i-1}{x_i}\sum_{j=0}^{k}\binom{x_i}{j}(-1)^{j}$$

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$$\times\left[\frac{(1+\alpha)(1+\beta)(1+\alpha+\beta)\left(2\left(1+\frac{r+j}{\lambda}\right)+\alpha+\beta\right)}{\left(1+\frac{r+j}{\lambda}\right)\left(1+\alpha+\frac{r+j}{\lambda}\right)\left(1+\beta+\frac{r+j}{\lambda}\right)\left(1+\alpha+\beta+\frac{r+j}{\lambda}\right)(2+\alpha+\beta)}\right]$$

$$\times\frac{b_r^{a_r}}{\Gamma(a_r)}r^{a_r-1}e^{-b_r r}\times\frac{b_\lambda^{a_\lambda}}{\Gamma(a_\lambda)}\lambda^{a_\lambda-1}e^{-b_\lambda\lambda}\times\frac{b_\alpha^{a_\alpha}}{\Gamma(a_\alpha)}\alpha^{a_\alpha-1}e^{-b_\alpha\alpha}\times\frac{b_\beta^{a_\beta}}{\Gamma(a_\beta)}\beta^{a_\beta-1}e^{-b_\beta\beta} \quad (8)$$

and the posterior distribution of r,λ,α,β was

$$\pi\left(r,\lambda,\alpha,\beta\middle|y_1,y_2,...,y_n\right)=\frac{f\left(y_1,y_2,...,y_n,r,\lambda,\alpha,\beta\right)}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty f\left(y_1,y_2,...,y_n,r,\lambda,\alpha,\beta\right)dr\,d\lambda\,d\alpha\,d\beta} \quad (9)$$

Therefore, the Bayesian estimator of r,λ,α,β under a squared error loss function can be achieved from expectation of posterior distribution were

$$\hat{r}_{Bayes}=E\left(r\middle|y_1,y_2,...,y_n\right)=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty r\pi\left(r,\lambda,\alpha,\beta\middle|y_1,y_2,...,y_n\right)dr\,d\lambda\,d\alpha\,d\beta$$

$$\hat{\lambda}_{Bayes}=E\left(\lambda\middle|y_1,y_2,...,y_n\right)=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \lambda\pi\left(r,\lambda,\alpha,\beta\middle|y_1,y_2,...,y_n\right)dr\,d\lambda\,d\alpha\,d\beta$$

$$\hat{\alpha}_{Bayes}=E\left(\alpha\middle|y_1,y_2,...,y_n\right)=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \alpha\pi\left(r,\lambda,\alpha,\beta\middle|y_1,y_2,...,y_n\right)dr\,d\lambda\,d\alpha\,d\beta$$

$$\hat{\beta}_{Bayes}=E\left(\beta\middle|y_1,y_2,...,y_n\right)=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \beta\pi\left(r,\lambda,\alpha,\beta\middle|y_1,y_2,...,y_n\right)dr\,d\lambda\,d\alpha\,d\beta$$

We can obtain estimated parameters r,λ,α,β by using the LaplacesDemon function in LaplacesDemon package in R program. In the next section, we will illustrate the application with a real data set with Metropolis-within-Gibbs algorithm, by R Program.

Application study

In studying its use, the NB-TWE distribution is applied to biological data sets and compared with a Poisson and NB distribution. The first three data sets are the European corn-borer data, which were presented in McGuire et al. (1957). The first data set provides information on the number of European corn-borers per plant with groups 0, 1, 2, 3, 4 and 5 having counts of 188, 83, 36, 14, 2 and 1. The second data set has counts of 907, 275, 88, 23, 3. The third data set is the distribution of Pyrausta nubilalis in 1937 by Beall (1940), with the

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number of insects in groups 0, 1, 2, 3, 4, and 5 having counts of 33, 12, 6, 3, 1 and 1. The results for fitting the distribution are shown in Tables 1 to 3.

Table 1. Goodness of fit test for the Poisson, NB and NB-TWE distributions when fitting European corn-borer data

Number of borers per plant X group 1	*Observed frequency*	*Expected value by fitting distribution*		
		Poisson	*NB*	*NB-TWE*
0	188	164.32	163.73	194.19
1	83	111.56	107.23	79.49
2	36	37.87	39.39	29.86
3	14	8.57	10.70	11.69
4	2	1.45	2.39	4.82
5	1	0.20	0.47	2.09
Parameters estimates		$\hat{\lambda}$ = 0.6789	$\hat{r}$ = 8.2054 $\hat{p}$ = 0.9202	$\hat{r}$ = 8.0329 $\hat{\lambda}$ = 14.7193 $\hat{\alpha}$ = 9.6251 $\hat{\beta}$ = 9.7089
Deviance		726.4856	719.4940	718.3184
K-S Test (p-value)		0.0731 (0.0627)	0.0749 (0.0527)	0.0191 (0.9998)
Cramer-von Mises-W2 (p-value)		0.7642 (0.0107)	0.7629 (0.0108)	0.0622 (0.5549)
Cramer-von Mises-A2 (p-value)		3.4351 (0.0123)	3.1092 (0.0178)	0.4000 (0.5439)

The number of borers per plant X group 1 shows that NB-TWE distribution gives the smallest value of K-S test (p-value is 0.9998) and Cramer-von Mises A2 test (p-value is 0.5439).

Table 2 shows that the NB-TWE distribution gives the smallest value of the K-S test (p-value is 1), and Cramer-von Mises A2 test (p-value is 0.5061).

The number of insects X in Table 3 shows that the NB-TWE distribution gives the smallest value of the K-S test (p-value is 1), and Cramer-von Mises A2 test (p-value is 0.9025).

The number of borers per plant X group 1-2 column, and number of insects X column in Table 3 are from the data sets that have an overdispersion problem, and are heavy-tailed with positively skewness. The results of fitting the different distributions are compared by using the K-S Test and Cramer-von Mises Test, we found that the data follows the NB-TWE distribution with the largest p-value and gives the smallest value of K-S Test and Cramer-von Mises Test. It also suggests that the NB-TWE distribution is the best fit for these data sets. Therefore, the NB-TWE distribution can be considered as an alternative model for count data that have an overdispersion problem and are heavy-tailed with positive skewness.

 doi: 10.14416/j.appsci.2019.03.001

Table 2. Goodness of fit test for the Poisson, NB and NB-TWE distribution when fitting European corn-borer data.

Number of borers per plant X group 2	*Observed frequency*	*Expected value by fitting distribution*		
		Poisson	*NB*	*NB-TWE*
0	907	854.11	883.45	918.05
1	275	356.15	308.50	268.35
2	88	74.25	80.40	75.33
3	23	10.32	18.58	22.71
4	3	1.08	4.02	7.39
Parameters estimates		$\hat{\lambda}$ = 0.4143	$\hat{r}$ = 2.0446 $\hat{p}$ = 0.8277	$\hat{r}$ = 6.2350 $\hat{\lambda}$ = 18.1648 $\hat{\alpha}$ = 9.2200 $\hat{\beta}$ = 9.5190
Deviance		2235.8843	2205.6814	2210.3294
K-S Test (p-value)		0.0408 (0.0267)	0.0182 (0.7856)	0.0085 (1.0000)
Cramer-von Mises-W2 (p-value)		1.1156 (0.0013)	0.2087 (0.1635)	0.0526 (0.5153)
Cramer-von Mises-A2 (p-value)		6.7564 (0.0039)	1.0785 (0.1831)	0.3862 (0.5061)

Table 3. Goodness of fit test for the poisson, NB and NB-TWE distribution when fitting *Pyrausta nublilalis* data.

Number of insects X	*Observed frequency*	*Expected value by fitting distribution*		
		Poisson	*NB*	*NB-TWE*
0	33	22.25	23.88	31.07
1	12	20.54	19.13	14.17
2	6	9.48	8.79	5.91
3	3	2.91	3.04	2.55
4	1	0.67	0.88	1.16
5	1	0.12	0.22	0.55
Parameters estimates		$\hat{\lambda}$ = 0.9228	$\hat{r}$ = 7.1675 $\hat{p}$ = 0.8868	$\hat{r}$ = 7.9328 $\hat{\lambda}$ = 12.1495 $\hat{\alpha}$ = 10.1997 $\hat{\beta}$ = 10.2329
Deviance		145.9365	140.5059	135.2861
K-S Test (p-value)		0.1919 (0.0324)	0.1629 (0.1024)	0.0345 (1.0000)
Cramer-von Mises-W2 (p-value)		0.8152 (0.0076)	0.5900 (0.0256)	0.0289 (0.8030)
Cramer-von Mises-A2 (p-value)		3.5561 (0.0116)	2.4875 (0.0389)	0.1121 (0.9025)

Conclusion

The negative binomial-two parameter weighted exponential (NB-TWE) is a discrete distribution that is obtained by mixing the NB distribution with the TWE distribution. This new distribution is heavy-tailed and is positively skewed. By fitting results, it shows that the NB-TWE distribution gives the smallest value of K-S Test and Cramer-von Mises Test.

From analyzing biological data for which the variance is significantly larger than the mean, and the data are heavy-tailed, we show the NB-TWE distribution provides a better fit than the Poisson and NB distributions. Consequently, the NB-TWE distribution is the best choice for biological data that have an overdispersion problem and are heavy-tailed and positively skewed.

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