On the output feedback control of discrete-valued input systems

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ABSTRACT

This paper considers an output feedback control for quantized feedback systems. Our controller focuses on high accuracy control performance for embedded devices with low-resolution AD/DA converters and networked systems with band-limited channels. The synthesis problem we address is the simultaneous synthesis of the nominal controller and the delta-sigma modulator (where the modulators are called the dynamic quantizers). For certain systems, we provide closed form and numerical solutions for the synthesis problem based on the invariant set analysis and the LMI technique. First, this paper proposes a synthesis condition that is recast as a set of matrix inequality conditions. The condition reduces to a tractable numerical optimization problem. Second, a closed form solution of optimal controller for the quantized feedback system is clarified within the invariant set framework. Third, we discuss the controller synthesis conditions which are characterized by the transmission zero property. Finally, to verify the validity of our method, numerical examples are presented and then the contributions related to the existing dynamic quantizer synthesis are clarified.

Keywords: Discrete-Valued Input, LMI, Output Feedback Control, Invariant Set Analysis, Simultaneous Synthesis

1. INTRODUCTION

Recently, one of the most remarkable control studies is the discrete-valued control problem. A number of analysis and synthesis methods for control systems including discrete-valued signal have been studied so far [1-14]. In the networked systems such as wireless control of mobile robots as shown in Fig.1, continuous-valued signals are quantized into discrete-valued signals and transmitted/received over communication channels. In this case, we need appropriate quantization methods to achieve some control performance requirements such as stabilization over communication channels and to characterize the minimum data rates for stabilization. Since the discrete-valued control theory can be applied not only to the networked control and but also to the various devices such as D/A or A/D converters, ON/OFF actuators and system biology, this control topic has been actively studied so far.

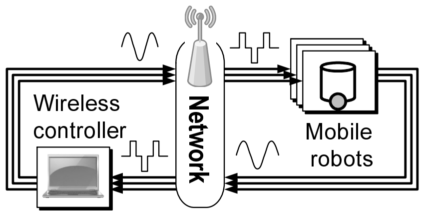
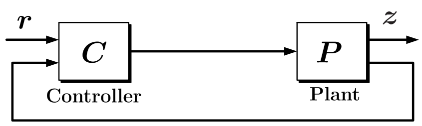
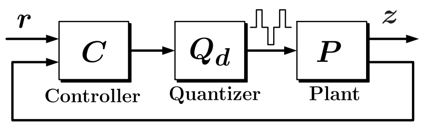


Fig.1: Networked control system



(a) Control system with continuous-valued input



(b) Control system with quantized-valued input

Fig.2: Two control systems

For the above challenging problem, references [7-14] have focused on optimality of the systems controlled by discrete-valued signals and provided an optimal dynamic quantizer for the discrete-valued control. The dynamic quantizer synthesis is the following statement: When a plant P and a controller C are given in the linear feedback system in Fig.2 (a), design a "dynamic" quantizer Q_d such that the system in Fig.2 (b) "optimally" approximates the usual system in Fig.2 (a) in the sense of the input-output relation. The dynamic quantizer formulation includes the delta-sigma modulator [15]. The optimal dynamic quantizer enables us to design the controller C in Fig.2 (b) based on the conventional linear control system theory [16-19].

The above design is two-step synthesis framework. A disadvantage of the existing approaches is the inherent suboptimality due to two-step synthesis of the nominal controller and the dynamic quantizer. On the other hand, simultaneous synthesis of the nomi-

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nal controller and the dynamic quantizer may achieve better performance than two-step synthesis. Then, this paper considers this challenging problem. As a first step, we focus on the synthesis of the controller with the delta-sigma modulation mechanism (where its modulation structure is the dynamic quantizer one) for feedback control systems with quantized inputs. Our approach is based on the invariant set analysis [16,20] and the linear matrix inequality (LMI) technique [17-19].

First, this paper proposes a controller analysis condition for quantized feedback systems. Second, we provide a simultaneous synthesis condition that is recast as a set of matrix inequalities. The condition reduces to a tractable numerical optimization problem. Third, a closed form solution of a controller for the quantized feedback system is clarified within the invariant set framework. In addition, we discuss the controller synthesis conditions which are characterized by the transmission zero property. Our method naturally extends to multiobjective control problems [16,18,19]. As an example, we then consider active control of pneumatic isolation table with on-off drive input [21] to verify the validity of our proposed method. Also, how our simultaneous synthesis relates to the existing dynamic quantizer synthesis is also discussed. A numerical example clarifies that our simultaneous synthesis is superior to the dynamic quantizer synthesis in terms of the control system order and the nominal controller design.

Notation: The set of $n \times m$ (positive) real matrices is denoted by $\mathbb{R}^{n\times m}$ ($\mathbb{R}_+^{n\times m}$). The set of $n \times m$ (positive) integer matrices is denoted by $\mathbb{N}^{n\times m}$ ($\mathbb{N}_+^{n\times m}$). The set of bounded sequences of p-dimensional vectors is denoted by ℓ_∞^p. $0_{n\times m}$ and I_m (or for simplicity of notation, 0 and I) denote the $n \times m$ zero matrix and the $m \times m$ identity matrix, respectively. For a matrix M, M^T, $\rho(M)$ and $\lambda_{\max}(M)$ denote its transpose, its spectrum radius and its maximum eigen value, respectively. For a matrix $M := \{M_{ij}\}$, $\mathrm{abs}(M)$ denotes the matrix composed of the absolute values of the elements, i.e., $\mathrm{abs}(M) := \{|M_{ij}|\}$. diag $(M_1, M_2, ..., M_m)$ denotes the $nm \times nm$ block diagonal matrix whose diagonal elements are M_1, M_2,...,$M_m \in \mathbb{R}^{n\times n}$. For a vector x, x_i is the i^{th} entry of x. For a symmetric matrix X, $X > 0$ ($X \geq 0$) means that X is positive (semi) definite. For a matrix X, $\|X\|_2$ denotes its 2-norms. For a vector x and a sequence of vectors $X := \{x_1, x_2, ...\}$, $\|x\|$ and $\|X\|$ denote their ∞-norms, respectively. Finally, we use the "packed" notation for transfer functions: $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) := C(\mathbf{z}I - A)^{-1}B + D$.

2. PRELIMINARIES

Consider the linear time invariant (LTI) discrete-time system given by

$$\xi(t+1) = \mathcal{A}\xi(t) + \mathcal{B}w(t) \tag{1}$$

where $\xi \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$ denote the state vector and disturbance input, respectively. We define the invariant set.

Definition 1: Define the invariant set of the system (1) to be a set $\mathcal{X}$ which satisfies

$$\xi \in \mathcal{X}, \quad w \in \mathcal{W} \quad \Rightarrow \quad \mathcal{A}\xi + \mathcal{B}w \in \mathcal{X}$$

where $\mathbb{W} := \{w \in \mathbb{R}^m : w^T w \leq 1\}$.

The analysis condition can be expressed in terms of matrix inequalities as summarized in the following proposition.

Proposition 1: [20] Consider the system (1). For a matrix $0 < \mathcal{P} \in \mathbb{R}^{n\times n}$, the ellipsoid $\mathbb{E}(\mathcal{P}) := \left\{\xi \in \mathbb{R}^n : \xi^T\mathcal{P}\xi \leq 1\right\}$ is an invariant set if and only if there exists a scalar $\alpha \in [0, 1 - \rho(\mathcal{A})^2]$ satisfying

$$\begin{bmatrix} \mathcal{A}^T\mathcal{P}\mathcal{A} - (1-\alpha)\mathcal{P} & \mathcal{A}^T\mathcal{P}\mathcal{B} \\ \mathcal{B}^T\mathcal{P}\mathcal{A} & \mathcal{B}^T\mathcal{P}\mathcal{B} - \alpha I_m \end{bmatrix} \leq 0. \tag{2}$$

The all ellipsoidal invariant sets are parameterized by Proposition 1. Also, $\mathbb{E}(\mathcal{P})$ allows us to approximate the reachable set from outside since the former covers the latter. Reference [20] considers the criterion $f(\mathcal{P})$ for the approximation of $\mathbb{E}(\mathcal{P})$ to the reachable set because the matrix $\mathcal{P}$ determines the ellipsoid. $f(\mathcal{P})$ has the monotonical decreasingness in the sense that its value for the set of inside is less than that of outside. When α is fixed in (2), reference [20] clarifies that the infimum of $f(\mathcal{P})$ does not change even if $\mathcal{P}$ is restricted to $\mathcal{P}(\alpha)$ given by

$$\mathcal{P}(\alpha)^{-1} = \sum_{k=0}^{\infty} \frac{1}{\alpha(1-\alpha)^k}\mathcal{A}^k\mathcal{B}\mathcal{B}^T(\mathcal{A}^T)^k \tag{3}$$

where $\alpha \in (0, 1-\rho(\mathcal{A})^2)$. Thus the criterion $f(\mathcal{P})$ can be replaced by $f(\mathcal{P}(\alpha))$ as well as the invariant sets in (2) can be parameterized by $\alpha \in (0, 1 - \rho(\mathcal{A})^2)$. Denote by $\xi(t, \xi(0), w)$ the state trajectory of the system (1) at the t-th time. For the set $\mathbb{E}(\mathcal{P})$ characterized by Proposition 1, the property

$$\lim_{t\to\infty} \inf_{\xi\in\mathbb{E}(\mathcal{P})} \|\xi(t, \xi(0), w) - \xi\| = 0 \tag{4}$$

also holds clearly (see [22]).

3. PROBLEM FORMULATION

Consider the quantized feedback system Σ_Q as shown Fig. 3, which consists of the LTI discrete-time plant P and the output feedback controller C. The plant P is given by

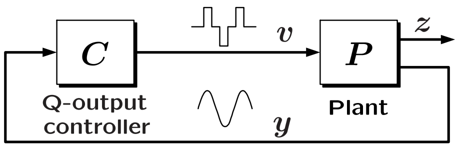

Fig.3: *Quantized feedback control system* Σ_Q

$$\begin{bmatrix} x(t+1) \\ z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} \tag{5}$$

where $x \in \mathbb{R}^{n_p}$, $z \in \mathbb{R}^p$, $v \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$ denote the state, the controlled output, the plant input and the measured output, respectively.

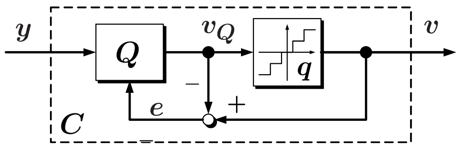

Fig.4: *Quantized output controller*

For the system P, we consider the quantized-output controller C such that its output signal v belongs to the discrete set on which each output takes values. In order to mitigate undesirable performance degradations caused by its discrete-valued output, the controller has delta-sigma modulation mechanism as shown in Fig. 4. The spatial resolution of the controller $v = C(y)$ is expressed by the static quantizer $q : \mathbb{R}^m \to d\mathbb{N}^m$ with the quantization interval $d \in \mathbb{R}$, i.e.,

$$v = q(v_Q)$$

and the discrete-time LTI filter Q is given by

$$v_Q = Q(\mathbf{z}) \begin{bmatrix} y \\ v - v_Q \end{bmatrix}, \quad Q(\mathbf{z}) := \begin{bmatrix} Q_1(\mathbf{z}) & Q_2(\mathbf{z}) \end{bmatrix}.$$

Let a state space realization of Q with the state vector $x_Q \in \mathbb{R}^{n_Q}$ be denoted by

$$\begin{bmatrix} x_Q(t+1) \\ v_Q(t) \end{bmatrix} = \begin{bmatrix} A_Q & B_{Q1} & B_{Q2} \\ C_Q & D_Q & 0 \end{bmatrix} \begin{bmatrix} x_Q(t) \\ y(t) \\ v(t) - v_Q(t) \end{bmatrix}. \tag{6}$$

The dynamic quantizer synthesis in [8-14] is to first design the nominal controller $Q_1(\mathbf{z})$ to achieve good performance without considering the quantization influence, and then add the modulator $Q_2(\mathbf{z})$ to minimize the quantization influence on the controlled output. In contrast, we consider the simultaneous synthesis problem of designing $Q_1(\mathbf{z})$ and $Q_2(\mathbf{z})$ at the same time. Note that q is of the nearest-neighbor type toward $-\infty$ with the quantization interval $d \in \mathbb{R}_+$ such as the midtread type quantizer in Fig. 5 and the initial state is given by $x_Q(0) = 0$ for the drift-free of C [8,9].

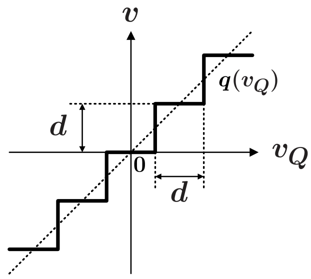

Fig.5: *Static quantizer*

For the closed-loop system in Fig. 3, the system P with the static quantizer q seen by the linear compensator Q can be described as the linear fractional transformation (LFT) of a generalized plant G:

$$\begin{bmatrix} x(t+1) \\ v_Q(t) \\ z(t) \\ y(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A & B & B \\ 0 & 0 & I \\ C_1 & 0 & 0 \\ C_2 & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \\ v_Q(t) \end{bmatrix}$$

and the quantization error q_e :

$$e = q_e(v_Q), \quad q_e(v_Q) := q(v_Q) - v_Q \tag{7}$$

where the signal $e \in [-d/2, d/2]^m$. The closed-loop system Σ_Q in Fig. 1 is described as an LFT of the quantization error $q_e(v_Q)$ and an LTI system Σ

$$e = q_e(v_Q), \quad \begin{bmatrix} v_Q \\ z \end{bmatrix} = \Sigma(\mathbf{z})e \tag{8}$$

where Σ is defined as a feedback connection of G and Q as shown in Fig. 6. Let the state space realization of Σ with the state vector $x_\Sigma \in \mathbb{R}^n$ be denoted by

$$\begin{bmatrix} x_\Sigma(t+1) \\ v_Q(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C}_1 & 0 \\ \mathcal{C}_2 & 0 \end{bmatrix} \begin{bmatrix} x_\Sigma(t) \\ e(t) \end{bmatrix}, \quad x_\Sigma := \begin{bmatrix} x \\ x_Q \end{bmatrix} \tag{9}$$

where $n := n_p + n_Q$.

We consider a control synthesis problem for the feedback control system with quantized actuators described above. For the system in Fig. 1, $z(t, x_0)$ denotes the output of z at the t-th time for the initial state $x_0 := x(0)$. In this case, this paper considers the following cost function:

$$J(Q) := \sup_{x_0 \in \mathbb{R}^{n_p}} \limsup_{t \to \infty} \|z(t, x_0)\|.$$

We consider a characterization of the cost function

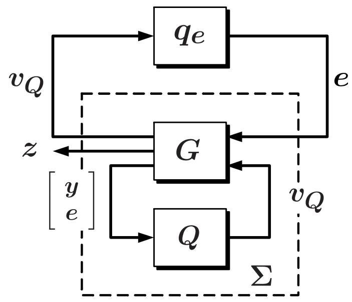

Fig.6: *Feedback system with quantization error*

$J(Q)$ in terms of invariant set analysis, and show that a feasible controller exists if a set of BMIs are solvable. Also, we define the stability of the quantized feedback system Σ_Q and the controller C as follows:

Definition 2: The quantized feedback system Σ_Q is said to be stable if the state (x, x_Q) is bounded for every initial state $x_0 \in \mathbb{R}^{n_p}$. The controller C is said to be stable if the state x_Q is bounded for every initial state $x_0 \in \mathbb{R}^{n_p}$.

The synthesis problem (**Q**) we address is the following: *For the quantized feedback system Σ_Q, suppose that the quantization interval $d \in \mathbb{R}_+$ and the performance level $\gamma \in \mathbb{R}_+$ are given. Characterize a stable quantized output controller C (i.e., find filter parameters $(n_Q, A_Q, B_{Q1}, B_{Q2}, C_Q, D_Q)$) achieving $J(Q) \le \gamma$ based on Proposition 1.*

From Proposition 1, the stability of the obtained controller C from Problem (**Q**) is guaranteed similar to the dynamic quantizer synthesis in [11, 13, 14]. Because of the quantization error caused by the static quantizer, the controlled output z of the system Σ_Q might not go to zero and might go unbounded no matter where it starts and no matter how long time passes. If the minimum value of γ is sufficient small, the controller minimizes the effect of the quantization error on the controlled output z in a neighborhood of the origin.

4. MAIN RESULT

4.1 Controller analysis

Suppose that the controller C to be analyzed is given. Define the set $\varepsilon := \{w \in \mathbb{R}^m : e = \frac{\sqrt{m}d}{2} w \text{ satisfies (7)}\}$ and rewrite system (9) as

$$\Sigma' : \begin{bmatrix} \xi(t+1) \\ v_p(t) \\ z_p(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C}_1 & 0 \\ \mathcal{C}_2 & 0 \end{bmatrix} \begin{bmatrix} \xi(t) \\ w(t) \end{bmatrix} \quad (10)$$

where

$$\xi := \frac{2}{\sqrt{m}d} x_\Sigma, \quad v_p := \frac{2}{\sqrt{m}d} v_Q, \quad z_p := \frac{2}{\sqrt{m}d} z. \quad (11)$$

The relation $\varepsilon \subseteq \mathbb{W}$ clearly holds since $e^T e \le \frac{md^2}{4}$ and the set $\mathbb{W}$ is an independent bounded disturbance without the relation (7). That is, the reachable set of Σ' with $w \in \varepsilon$ is no larger than that of Σ' with the disturbance $w \in \mathbb{W}$. Then, we can use the reachable set to estimate the influences of the quantization error and the invariant set to characterize the cost function $J(Q)$.

The ellipsoidal invariant set $\mathbb{E}(\mathcal{P})$ can be parameterized by Proposition 1, while covering the reachable set from the outside for (2,1) block of the system Σ'. In addition, if there exists the set $\mathbb{E}(\mathcal{P})$, there exists a scalar $\gamma \in \mathbb{R}_+$ satisfying

$$\max_i \sup_{\xi \in \mathbb{E}(\mathcal{P})} |c_i^T \xi| = \gamma \quad \Leftarrow \quad \begin{bmatrix} \mathcal{P} & \mathcal{C}_2^T \\ \mathcal{C}_2 & \gamma^2 I_p \end{bmatrix} \ge 0 \quad (12)$$

where c_i^T is the i-th entry of $\mathcal{C}_2$ (see [11, 16]). Then, from the property (4) of the invariant set and (11), the performance level γ in (12) satisfies

$$J(Q) \le \gamma \frac{\sqrt{m}d}{2}. \quad (13)$$

In analysis, it is appropriate to treat $\mathcal{P}$ as a variable and search for $\mathcal{P}$ minimizing the performance level γ. For Proposition 1, we have the optimization problem (**Aop**):

$$\min_{\mathcal{P}>0, 1-\rho(\mathcal{A})^2>\alpha>0, \gamma>0} \gamma \quad \text{s.t.} \quad (2) \text{ and } (12).$$

If Problem (**Aop**) is feasible, property (4) is satisfied for the system Σ'. In other words, the state ξ of Σ' is bounded for the disturbance $e \in \varepsilon$ and the initial state $\frac{\sqrt{m}d}{2} x_0$, so the state $\frac{\sqrt{m}d}{2} x_Q$ is also bounded. From the relation between Σ_Q and Σ', we therefore have the following lemma of stability.

Lemma 1: The quantized feedback control system Σ_Q is stable if Problem (**Aop**) is feasible. The controller C is stable if Problem (**Aop**) is feasible.

Focusing on the left side of (12), we see that γ is corresponding to the criterion $f(\mathcal{P}(\alpha))$. From the parameterization $\mathcal{P}(\alpha)$ in (3), the infimum of γ can be expressed by the following lemma [13, 14].

Lemma 2: For the feedback system (8), suppose that the quantization interval $d \in \mathbb{R}_+$ is given. Consider Problem (**Aop**). The infimum of γ is given by

$$\inf \gamma = \inf_\alpha \frac{\lambda_\alpha}{\sqrt{\alpha}}, \quad 0 < \alpha < 1 - \rho(\mathcal{A})^2, \quad (14)$$

$$\lambda_\alpha := \sqrt{\lambda_{\max}\left(\sum_{k=0}^{\infty} \frac{1}{(1-\alpha)^k} \mathcal{C}_2 \mathcal{A}^k \mathcal{B}\mathcal{B}^T (\mathcal{A}^T)^k \mathcal{C}_2^T\right)}.$$

Proof: Define $\gamma(\alpha)$ which is obtained from Problem (**Aop**) for the fixed α. Applying schur complement to (12) yields

$$(12) \Leftrightarrow \gamma(\alpha)^2 I_p - \mathcal{C}_2 \mathcal{P}(\alpha)^{-1} \mathcal{C}_2^T \ge 0.$$

Substituting (3) results in

$$\gamma(\alpha)^2 I_q \geq \frac{1}{\alpha}\sum_{k=0}^{\infty}\frac{1}{(1-\alpha)^k}\mathcal{C}_2\mathcal{A}^k\mathcal{B}\mathcal{B}^T(\mathcal{A}^T)^k\mathcal{C}_2^T.$$

Hence, the infimum of γ is given by (14). ■

4.2 Controller synthesis

Problem **(Aop)** suggests that Problem (**Q**) reduces to the following non-convex optimization problem (**S**):

$$\min_{\mathcal{P},A_Q,B_{Q_1},B_{Q_2},C_Q,D_Q,\alpha,\gamma} \gamma \ \text{ s.t. } \ (2) \ \text{ and } \ (12).$$

From Lemma 1, if Problem (**S**) is feasible, the obtained controller C is stable and the resulting quantized feedback control system Σ_Q is stabilized. In addition, Problem (**S**) reduces to a tractable matrix inequality problem as summarized in the following theorem.

Theorem 1: For the feedback system (8), suppose that the quantization interval $d \in \mathbb{R}_+$ and the performance level $\gamma \in \mathbb{R}_+$ are given. For a scalar $\alpha \in (0,1)$, there exists a stable controller C achieving (13) if one of the following equivalent statements holds.
(i) There exist a matrix $0 < \mathcal{P} \in \mathbb{R}^{n\times n}$ and a controller C satisfying (2) and (12).
(ii) There exist matrices $0 < X \in \mathbb{R}^{n_p\times n_p}$, $0 < Y \in \mathbb{R}^{n_p\times n_p}$, $F \in \mathbb{R}^{m\times n_p}$, $L \in \mathbb{R}^{n_p\times m}$, $W \in \mathbb{R}^{n_p\times n_p}$, $M \in \mathbb{R}^{m\times m}$ and $U \in \mathbb{R}^{n_p\times m}$ satisfying

$$\begin{bmatrix} (1-\alpha)\Xi_{\mathsf{P}} & 0 & \Xi_{\mathsf{A}}^T \\ 0 & \alpha I_m & \Xi_{\mathsf{B}}^T \\ \Xi_{\mathsf{A}} & \Xi_{\mathsf{B}} & \Xi_{\mathsf{P}} \end{bmatrix} \geq 0, \quad \begin{bmatrix} \Xi_{\mathsf{P}} & \Xi_{\mathsf{C}}^T \\ \Xi_{\mathsf{C}} & \gamma^2 I_p \end{bmatrix} \geq 0 \quad (15)$$

where

$$\Xi_{\mathsf{P}} := \begin{bmatrix} X & I \\ I & Y \end{bmatrix}, \quad \Xi_{\mathsf{A}} := \begin{bmatrix} XA + LC_2 & W \\ A + BMC_2 & AY + BF \end{bmatrix},$$
$$\Xi_{\mathsf{B}} := \begin{bmatrix} U \\ B \end{bmatrix}, \qquad \Xi_{\mathsf{C}} := \begin{bmatrix} C_1 & C_1 Y \end{bmatrix}.$$

One such controller parameter ($n_Q = n_p$) is given by

$$\begin{bmatrix} A_Q & B_{Q1} & B_{Q2} \\ C_Q & D_Q & 0 \end{bmatrix} = \begin{bmatrix} Z & XBc \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} W - XAY & L & U - XB \\ F & M & 0 \end{bmatrix} \begin{bmatrix} -Y & 0 & 0 \\ C_2 Y & I & 0 \\ 0 & 0 & I \end{bmatrix}^{-1} \quad (16)$$

where $Z = X - Y^{-1}$.

Proof: Inequality (2) can be rewritten as

$$\begin{bmatrix} (1-\alpha)\mathcal{P} & 0 & \mathcal{A}^T\mathcal{P} \\ 0 & \alpha I_m & \mathcal{B}^T\mathcal{P} \\ \mathcal{P}\mathcal{A} & \mathcal{P}\mathcal{B} & \mathcal{P} \end{bmatrix} \geq 0. \quad (17)$$

For the full order case $n_Q = n_p$, an appropriate choice of the controller state coordinates then allows us to assume, without loss of generality, that $\mathcal{P}$ has the following special structure [19].

$$\mathcal{P} = \begin{bmatrix} X & Z \\ Z & Z \end{bmatrix}. \quad (18)$$

Note that the matrices $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}_2$ of (17) are given by

$$\mathcal{A} := \begin{bmatrix} A + BD_Q C_2 & BC_Q \\ B_{Q1} C_2 & A_Q \end{bmatrix}, \quad \mathcal{B} := \begin{bmatrix} B \\ B_{Q2} \end{bmatrix},$$
$$\mathcal{C}_2 := \begin{bmatrix} C_1 & 0 \end{bmatrix}.$$

Introducing the change of variables

$$\begin{bmatrix} W & L & U \\ F & M & 0 \end{bmatrix} := \begin{bmatrix} XAY & 0 & XB \\ 0 & & 0 \end{bmatrix} + \begin{bmatrix} Z & XB \\ 0 & I_m \end{bmatrix} \begin{bmatrix} A_Q & B_{Q1} & B_{Q2} \\ C_Q & D_Q & 0 \end{bmatrix} \begin{bmatrix} -Y & 0 & 0 \\ C_2 Y & I_m & 0 \\ 0 & 0 & I_m \end{bmatrix},$$
$$T := \begin{bmatrix} I_{n_p} & 0 \\ Y & -Y \end{bmatrix}, \quad (19)$$

the congruence transformation of the condition in (17) and (12) by $\mathrm{diag}\,(T, I_m, T)$ and $\mathrm{diag}\,(T, I_p)$ yields (15). Inequality (15) implies $\mathcal{P} \geq 0$, but not $\mathcal{P} > 0$. However, existence of positive definite $\mathcal{P}$ in statement (i) can be implied by slightly perturbing X and Y by εI with a sufficiently small scalar $\varepsilon \in \mathbb{R}_+$ [23]. Finally, the controller formula in (16) is obtained by solving the above equation for the controller parameters. ■

For the controller synthesis problem minimizing γ of (13), we have the optimization problem (**Sop**):

$$\min_{X>0,Y>0,F,L,W,U,M,1>\alpha>0,\gamma>0} \gamma \quad \text{s.t.} \quad (15).$$

In synthesis, the parameters $(A_Q, B_{Q1}, B_{Q2}, C_Q, D_Q)$ to be designed lead to $\alpha \in (0,1)$. When α is fixed, the conditions in Theorem 1 are linear matrix inequalities (LMIs) in terms of the other variables. Using standard LMI software in combination with the line search of α for **(Sop)**, we can obtain a controller, numerically.

Under some circumstances, Proposition 1 gives a controller which is expressed by the given plant parameters. The following theorem denotes this fact.

Theorem 2: Suppose that the matrix $C_1 A^\tau B$ is non-singular for the smallest integer $\tau \in \{0\} \cup \mathbb{N}_+$ satisfying $C_1 A^\tau B \neq 0$. Consider the controller C_{op} given by

$$\begin{cases} x_Q(t+1) &= (A - BMC_2 + LC_2 + BF_{op})x_Q(t) \\ &\quad +(BM - L)y(t) + B(v(t) - v_Q(t)), \\ v(t) &= q((F_{op} - MC_2)x_Q(t) + My(t)), \\ F_{op} &:= -(C_1 A^\tau B)^{-1} C_1 A^{\tau+1} \end{cases} \quad (20)$$

where matrices $M \in \mathbb{R}^{m\times m}$ and $L \in \mathbb{R}^{n_p\times m}$ are free parameters. If and only if the matrices $A + BF_{op}$ and $A + LC_2$ are stable, there exist $\mathcal{P} > 0$ and $\alpha \in (0, 1 - \rho(\Pi_A)^2)$ satisfying (2) for C_{op} and it's achievable infimum of $\gamma \in \mathbb{R}_+$ is

$$\inf \gamma = \frac{1}{\sqrt{\alpha_{op}(1-\alpha_{op})^{\tau}}}\|C_1A^{\tau}B\|_2, \qquad (21)$$

$$\alpha_{op} = \min\left\{\frac{1}{\tau+1}, 1-\rho(\Pi_A)^2\right\}$$

where the matrix Π_A is given by

$$\Pi_A := \begin{bmatrix} A+BF_{op} & -B(F_{op}-MC_2) \\ 0 & A+LC_2 \end{bmatrix}.$$

Proof: Consider the quantized feedback control system Σ_Q with (20) and define the following matrices:

$$\Gamma := \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}, \quad \Pi_B := \begin{bmatrix} B \\ 0 \end{bmatrix}.$$

In this case, the following relations hold.

$$\mathcal{A} = \begin{bmatrix} A+BMC_2 & B(F_{op}-MC_2) \\ (BM-L)C_2 & A-BMC_2+LC_2+BF_{op} \end{bmatrix}$$
$$= \Gamma\Pi_A\Gamma,$$
$$\mathcal{B} = \begin{bmatrix} B \\ B \end{bmatrix} = \Gamma\Pi_B, \quad \mathcal{C}_2 = \begin{bmatrix} C_1 & 0 \end{bmatrix} = \mathcal{C}_2\Gamma.$$

($\rightarrow$) We first show that there exist $\mathcal{P} > 0$ and $\alpha \in (0, 1-\rho(\mathcal{A})^2)$ satisfying (2) for the controller C_{op}. Using the fact that $\mathcal{P}$ of inequality (3) is parameterized by (3), we construct the matrix $\mathcal{P}$ given by

$$\mathcal{P} = \sum_{k=0}^{\infty}\frac{1}{\alpha(1-\alpha)^k}\mathcal{A}^k\mathcal{B}\mathcal{B}^T(\mathcal{A}^T)^k$$
$$= \sum_{k=0}^{\infty}\frac{1}{\alpha(1-\alpha)^k}(\Gamma\Pi_A\Gamma)^k\Gamma\Pi_B\Pi_B^T\Gamma^T(\Gamma^T\Pi_A^T\Gamma^T)^k$$
$$= \Gamma\sum_{k=0}^{\infty}\left(\frac{\Pi_A}{\sqrt{1-\alpha}}\right)^k\frac{\Pi_B}{\sqrt{\alpha}}\frac{\Pi_B^T}{\sqrt{\alpha}}\left(\frac{\Pi_A^T}{\sqrt{1-\alpha}}\right)^k\Gamma^T.$$

Defining $\mathcal{Q} = \Gamma\mathcal{P}\Gamma^T$ ($\Gamma = \Gamma^{-1}$, $\Gamma\Gamma = I$), we consider the following Lyapunov inequality:

$$\frac{\Pi_A}{\sqrt{1-\alpha}}\mathcal{Q}\frac{\Pi_A^T}{\sqrt{1-\alpha}} - \mathcal{Q} + \frac{\Pi_B}{\sqrt{\alpha}}\frac{\Pi_B^T}{\sqrt{\alpha}} \leq 0. \qquad (22)$$

The matrix $\frac{\Pi_A}{\sqrt{1-\alpha}}$ is stable for $\alpha \in (0, 1-\rho(\mathcal{A})^2)$ ($\rho(\mathcal{A}) = \rho(\Pi_A)$). Then, inequality (22) implies existence of $\mathcal{Q} \geq 0$, but not $\mathcal{Q} > 0$. However, existence of positive definite $\mathcal{P}$ can be implied by slightly perturbing $\mathcal{Q}$ by εI with a sufficiently small scalar $\varepsilon \in \mathbb{R}_+$. That is, such $\mathcal{P} = (\mathcal{P} + \varepsilon I)^{-1} > 0$ satisfies (2) for α guaranteeing the stability of $\frac{\Pi_A}{\sqrt{1-\alpha}}$. Therefore, there exist $\mathcal{P} > 0$ and $\alpha \in (0, 1-\rho(\mathcal{A})^2)$ satisfying (2) for the controller C_{op}.

($\leftarrow$) The congruence transformation of the condition in (2) by $\mathsf{diag}(\Gamma^{-\mathrm{T}}, I_m)$ yields

$$\begin{bmatrix} \Pi_A^T\mathcal{P}_\Gamma\Pi_A - (1-\alpha)\mathcal{P}_\Gamma & \Pi_A^T\mathcal{P}_\Gamma\Pi_B \\ \Pi_B^T\mathcal{P}_\Gamma\Pi_A & \Pi_B^T\mathcal{P}_\Gamma\Pi_B - \alpha I_m \end{bmatrix} \leq 0$$

where $\mathcal{P}_\Gamma = \Gamma^T\mathcal{P}\Gamma$. From (1,1) block of the above inequality and the positivity of α, the following inequality

$$\Pi_A^{T\mathcal{P}_\Gamma\Pi_A - \mathcal{P}_\Gamma \leq \alpha\mathcal{P}_\Gamma < 0}$$

holds. The above inequality ensures that Π_A is stable, that is, $A + BF_{op}$ and $A + LC_2$ are stable.

Next, we obtain

$$\mathcal{C}_2\mathcal{A}^k\mathcal{B} = \mathcal{C}_2\Gamma\Pi_A^k\Gamma\mathcal{B} = \mathcal{C}_2\Pi_A^k\Pi_B$$
$$= C_1(A - B(C_1A^{\tau}B)^{-1}C_1A^{\tau+1})^kB$$
$$= \begin{cases} 0 & k \leq \tau-1 \\ C_1A^{\tau}B & k=\tau \\ 0 & k \geq \tau+1 \end{cases}$$

where the last relation is obtained from the assumption. From Lemma 2, we obtain

$$\inf \gamma(\alpha)^2 = \inf_{\alpha}\mu(\alpha)\lambda_{\max}\left(C_1A^{\tau}B(C_1A^{\tau}B)^T\right),$$
$$\mu(\alpha) := \frac{1}{\alpha(1-\alpha)^{\tau}} > 0, \quad 0 < \alpha < 1-\rho(\Pi_A)^2.$$

For the range $\alpha \in (0, 1-\rho(\Pi_A)^2)$, $\mu(\alpha)$ is strictly convex as follows:

$$\frac{d\mu(\alpha)}{d\alpha} = \frac{(\tau+1)\alpha-1}{\alpha^2(1-\alpha)^{\tau+1}},$$
$$\frac{d^2\mu(\alpha)}{d\alpha^2} = \frac{(\tau+2)((\tau+1)\alpha-1)^2+\tau}{(\tau+1)\alpha^3(1-\alpha)^{\tau+2}} > 0.$$

Therefore, we have

$$\inf_{\alpha}\mu(\alpha) = \mu(\alpha_{op}), \quad \alpha_{op} = \min\left\{\frac{1}{\tau+1}, 1-\rho(\Pi_A)^2\right\}.$$

Note that $\|X\|_2 = \sqrt{\lambda_{\max}(XX^T)} = \sqrt{\lambda_{\max}(X^TX)}$. Then, the achievable performance of (20) is given by (21). ■

From Theorems 1 and 2, we also see that C_{op} is one of the controllers obtained from (16). If the infimum (or minimum) of γ obtained from (**Sop**) is equivalent to (21), therefore, the controller form is parameterized by C_{op} in (20). In this case, the controller C_{op} is optimal for the quantized feedback system Σ_Q in the sense that the upperbound of the cost function γ is minimized.

The structure of (20) is explained as follows. Consider the quantized feedback system Σ' with the optimal controller C_{op}. The resulting matrices $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}_{\in}$ are given by the proof of Theorem 2 and $\mathcal{C}_{\infty}$ is given by

$$\mathcal{C}_1 = \left[\begin{array}{cc} MC & F_{op} - MC_2 \end{array} \right].$$

From the congruence transformation matrix Γ, the transfer function of the system Σ' in (10) is equivalent to the following system with the state $\xi^\dagger := \Gamma\xi$:

$$\begin{aligned}
\left[\begin{array}{c} v_p \\ z_p \end{array} \right] &= \Sigma^\dagger(\mathbf{z})w = \left[\begin{array}{c} \Sigma_1^\dagger(\mathbf{z}) \\ \Sigma_2^\dagger(\mathbf{z}) \end{array} \right] w, \\
\xi^\dagger &:= \left[\begin{array}{c} \frac{2}{\sqrt{m}d}x \\ \frac{2}{\sqrt{m}d}(x - x_Q) \end{array} \right], \\
\Sigma^\dagger(\mathbf{z}) &:= \left(\begin{array}{c|c} \Pi_A & \Pi_B \\ \hline \Pi_C & 0 \\ \mathcal{C}_2 & 0 \end{array} \right), \ \Pi_C := \left[\begin{array}{cc} F_{op} & -F_{op} + MC_2 \end{array} \right].
\end{aligned}$$

This is because the following relation holds.

$$\Sigma^\dagger(\mathbf{z}) = \left(\begin{array}{c|c} \Gamma\Pi_A\Gamma & \Gamma\Pi_B \\ \hline \Pi_C\Gamma & 0 \\ \mathcal{C}_2\Gamma & 0 \end{array} \right) = \left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C}_1 & 0 \\ \mathcal{C}_2 & 0 \end{array} \right) = \Sigma'(\mathbf{z}).$$

The structure of the system $\Sigma^\dagger$ is observer based control, so we see that a separation principle (it is well known as a principle of separation of estimation and control) for the controller (20) holds. The influence of the quantization error on the plant state x is estimated from the measured output y and its estimation information is stored in x_Q. Also, the term $-(C_1A^\tau B)^{-1}CA^{\tau+1}x_Q(t)$ tries to cancel the influence of $z_p(t+\tau+1)$ (equivalently, $z(t+\tau+1)$). This idea closely matches that of the disturbance observer [24] in the discrete-time domain.

Next, we consider the stability condition of $A + BF_{op}$ and $A + LC_2$. We first see that there exist matrices F and L such that $A + BF$ and $A + LC_2$ are stable if and only if the triple (A, B, C_2) is stabilizable and detectable. We next focus on the following relation:

$$\begin{aligned}
v &= v_Q + e = \frac{\sqrt{m}d}{2}(v_p + w) \\
&= \frac{\sqrt{m}d}{2}\left(\Sigma_1^\dagger(\mathbf{z}) + I\right) w = \Sigma_{ev}(\mathbf{z})e, \\
\Sigma_{ev}(\mathbf{z}) &:= \left(\begin{array}{c|c} \Pi_A & \Pi_B \\ \hline \Pi_C & I \end{array} \right).
\end{aligned}$$

Note that $\Sigma_{ev}(\mathbf{z})$ is the transfer function from the quantization error e to the discrete-valued input v and has the following realization property:

$$\Sigma_{ev}(\mathbf{z}) = \left(\begin{array}{c|c} A + BF_{op} & B \\ \hline F_{op} & I \end{array} \right). \qquad (23)$$

We denote by $\Sigma_{ev}^{inv}(\mathbf{z})$ the inverse system of $\Sigma_{ev}(\mathbf{z})$ and $\Sigma_{ev}^{inv}(\mathbf{z})$ is given by

$$\Sigma_{ev}^{inv}(\mathbf{z}) = \left(\begin{array}{c|c} A & -B \\ \hline F_{op} & I \end{array} \right)$$

which has the following relation

$$\begin{aligned}
\Sigma_{ev}^{inv}(\mathbf{z}) &= \sum_{k=0}^{\infty} \left(\frac{-F_{op}A^kB}{\mathbf{z}}^{k+1} \right) + I \\
&= (C_1A^\tau B)^{-1}\mathbf{z}^{\tau+1} \left(\sum_{k=\tau+1}^{\infty} \left(\frac{C_1A^kB}{\mathbf{z}}^{k+1} \right) \right. \\
&\qquad \left. + \frac{C_1A^\tau B}{\mathbf{z}^{\tau+1}} \right) \\
&= (C_1A^\tau B)^{-1}\mathbf{z}^{\tau+1} \sum_{k=0}^{\infty} \left(\frac{C_1A^kB}{\mathbf{z}^{k+1}} \right) \\
&= (C_1A^\tau B)^{-1}\mathbf{z}^{\tau+1}P_z(\mathbf{z}). \qquad (24)
\end{aligned}$$

Note that $C_1B = C_1AB = \cdots = C_1A^{\tau-1}B = 0$. $P_z(\mathbf{z})$ is the transfer function of P from v to z. From (24), so $\Sigma_{ev}(\mathbf{z})$ is the inverse system of $(C_1A^\tau B)^{-1}\mathbf{z}^{\tau+1}P_z(\mathbf{z})$. Then, the transmission poles of $\Sigma_{ev}(\mathbf{z})$ are equal to the transmission zeros of $P_z(\mathbf{z})$ and $\mathbf{z} = 0$ with the multiplicity $\tau + 1$. If $\Sigma_{ev}(\mathbf{z})$ has no pole-zero cancellation (23) is the minimal realization of $\Sigma_{ev}(\mathbf{z})$), the eigenvalues of $A + BF_{op}$ consists of the transmission zeros of $P_z(\mathbf{z})$ and $\mathbf{z} = 0$ with the multiplicity $\tau + 1$. Then we obtain the stability condition of the controller (20) as summarized in the following theorem.

Theorem 3: Suppose that the triple (A, B, C_2) is stabilizable and detectable. The controller C_{op} of Theorem 2 is stable if and only if the all transmission zeros of P are stable (i.e., the system P is minimum phase).

Proof: From Lemma 1, the controller C_{op} of Theorem 2 is stable. Also, $A + BF_{op}$ is stable if and only if the all transmission zeros of P are stable. From Theorem 2, therefore, we see that the statement of Theorem 3 holds. ■

To verify the validity of Theorem 3, consider the stable and unstable cases of C_{op}. Here, P is the discrete-time plant obtained form the following continuous-time one:

$$\dot{x} = \left[\begin{array}{cc} -1 & -1 \\ 0 & 1 \end{array} \right] x + \left[\begin{array}{c} 1 \\ 1 \end{array} \right] v, \quad z = C_1x, \quad y = \left[\begin{array}{cc} 1 & 10 \end{array} \right] x$$

and the zero-order hold with the sampling period $h = 0.1$. The quantization interval is $d = 1$.

Figure 7 shows the simulation result on the time responses of the system Σ_Q with the controller C_{op} for the following setting:

$$C_1 = \left[\begin{array}{cc} 3 & -7 \end{array} \right] \qquad (25)$$

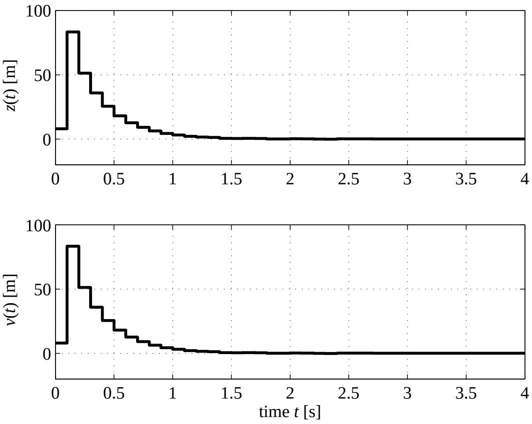

Fig.7: *Simulation results for C_{op} with (25)*

where $x_0 = [\ 5 \quad 1\ ]^T$ and $L = [\ 1.005 \quad -0.234\ ]^T$ and $M = -11.13$. In this case, the achievable performance given by (**Sop**) is 0.336 and the value of (21) is also 0.336. We see that the output behavior of Σ_Q with C_{op} is stable.

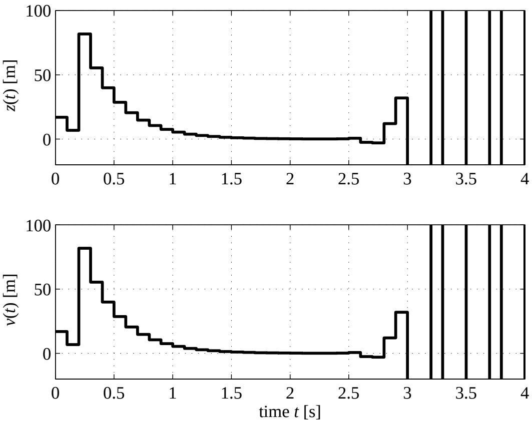

Fig.8: *Simulation results for C_{op} with (26)*

Instead of (25), we next consider the case C_1 is given as follows

$$C_1 = [\ 4 \quad -3\ ]. \tag{26}$$

Figure 8 shows the simulation result on the time responses of Σ_Q with C_{op} in the same fashion. We see that v and z diverge, thus the former controller is stable and the latter is unstable. The transmission zeros of P with (25) are $\{0.721, 0.930\}$ (0.721 is for $P_z(\mathbf{z})$), so the transmission poles of $\Sigma_{ev}(\mathbf{z})$ are $\{0, 0.721\}$. The transmission zeros of P with (26) are $\{0.930, 3.440\}$ (3.440 is for $P_z(\mathbf{z})$), so the transmission poles of $\Sigma_{ev}(\mathbf{z})$ are $\{0, 3.440\}$. Therefore, we see that Theorem 3 holds. For the non-minimum phase P, we can utilize (**Sop**). Figure 9 shows the simulation result on the time responses of Σ_Q with (26) and C obtained from (**Sop**) in the same fashion. We see that the output behavior of Σ_Q with C is stable, while that of Σ_Q with C_{op} is unstable.

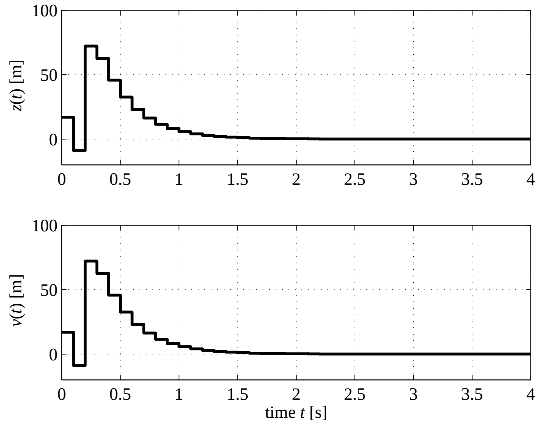

Fig.9: *Simulation results for* (**Sop**) *with (26)*

From Theorems 2 and 3, we conclude that the invariant analysis framework provides an optimal stable controller, which is characterized by the transmission zero of the plant. Our method focuses on the upper bound of the cost function $J(Q)$, so it is not clear whether the controller (20) optimizes $J(Q)$ in itself. To clarify this, we need ℓ_1 optimization technique which is presented for the optimal dynamic synthesis [8, 9] (a discussion omitted here due to space considerations). To clarify ℓ_1 optimality of the controller C_{op} is a future task.

5. DISCUSSION

5.1 Multiobjective control

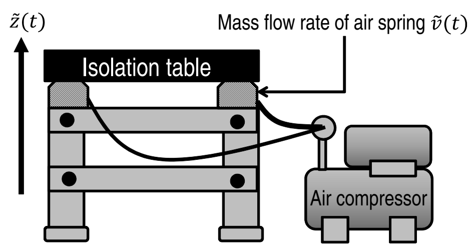

Fig.10: *Pneumatic isolation table*

Our method in Theorem 1 naturally extends to multiobjective control problems as shown in [16, 18, 19]. As an example, consider active control of pneumatic isolation table with on-off drive input [21] as shown in Fig. 10. The linearized continuous-time model is given by

$$\begin{bmatrix} \dot{\tilde{x}} \\ \tilde{z} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_{c1} & 0 \\ C_{c2} & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{v}(t-T_d) \end{bmatrix},$$

$$A_c = \begin{bmatrix} 0.00 & 1.00 & 0.00 & 0.00 \\ -686.76 & -1.88 & 0.00 & 0.00 \\ 0.00 & -1.96 & -194.69 & 194.69 \\ 0.00 & 0.00 & 26.67 & -26.67 \end{bmatrix},$$

$$B_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 8.32\times 10^8 \end{bmatrix}, \quad \begin{matrix} C_{c1} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \\ C_{c2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{matrix}$$

The controlled output $\tilde{z}$ [m] is the displacement of the isolation table. The measured output $\tilde{y}$ includes the displacement of the isolation table and the buffer tank pressure deviation. $\tilde{v}(t)$ [kg/s] is the on-off control input (mass flow rate) which is given by

$$\tilde{v} = q_\phi(\sigma) = \begin{cases} 2d & \sigma \geq 3d/2 \\ d & d/2 \leq \sigma < 3d/2 \\ -d & -3d/2 < \sigma \leq -d/2 \\ -2d & \sigma \leq -3d/2 \\ 0 & \text{others} \end{cases}.$$

$T_d \in \mathbb{R}_+$ is the time-delay of mass flow rate. To control the vertical displacement $\tilde{z}$ by on-off control under the input time delay, we consider the discretized system with a sampling time $h \in \mathbb{R}_+$ and zero-order hold as follows

$$\begin{bmatrix} x_d(t+1) \\ z(t) \\ y_d(t) \end{bmatrix} = \begin{bmatrix} A_d & B_d \\ C_{c1} & 0 \\ C_{c2} & 0 \end{bmatrix} \begin{bmatrix} x_d(t) \\ v(t-T_d/h) \end{bmatrix}, \quad (27)$$

$$A_d = \exp(A_c h), \quad B_d = \int_0^h \exp(A_c\tau) B_c d\tau$$

where the vectors x_d, z, y_d and v are the discretized vectors of $\tilde{x}$, $\tilde{z}$, $\tilde{y}$ and $\tilde{v}$. When T_d/h is a positive integer number, (27) is thus recast as the system (5) where

$$A = \begin{bmatrix} A_d & B_d & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix},$$

$$C_1 = \begin{bmatrix} C_{c1} & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} C_{c2} & 0 \\ 0 & I \end{bmatrix}.$$

In this case, the state $x(t)$ and the measured output $y(t)$ include the past input such as $x(t) = [x_d^T(t)\ v(t-T_d/h)^T\ v(t-(T_d/h-1))^T\ \ldots\ v(t-1)^T]^T$ and $y(t) = [y_d^T(t)\ v(t-T_d/h)^T\ v(t-(T_d/h-1))^T\ \ldots\ v(t-1)^T]^T$. This idea is utilized in [21].

To consider the transient performance of the nominal controller $Q_1(\mathbf{z})$ of (6), we introduce the following condition:

$$\mathcal{A}^T\mathcal{P}\mathcal{A} - \mathcal{P} + \frac{1}{\beta}\mathcal{C}_3^T\mathcal{C}_3 < 0, \quad (28)$$

$$\mathcal{C}_3 := \begin{bmatrix} \widehat{Q}^{\frac{1}{2}}C_1 & 0 \\ \widehat{R}^{\frac{1}{2}}D_Q C_2 & \widehat{R}^{\frac{1}{2}}C_Q \end{bmatrix}.$$

We assume $v = v_Q$ ($Q_2(\mathbf{z})$ does not operate). For the given scalar $\beta > 0$ and the controller $Q_1(\mathbf{z})$, if there exists a matrix $\mathcal{P} > 0$ satisfying (28), the following quadratic performance is achieved:

$$J_{cost} = \sum_{t=0}^{\infty}\left\{z(t)^T\widehat{Q}z(t) + v_Q(t)^T\widehat{R}v_Q(t)\right\} \leq \beta x_\Sigma(0)^T\mathcal{P}x_\Sigma(0) \quad (29)$$

where $\widehat{Q} \geq 0$ and $\widehat{R} \geq 0$ are the weight matrices. Regarding the above fact, please see Appendix A. To improve the transient vibration suppression performance, we have the optimization problem (**Mop**):

$$\min_{X,Y,F,L,W,U,1>\alpha>0,\gamma,\beta,\mu} \gamma + \mu\beta \text{ s.t. (15) and}$$

$$\begin{bmatrix} \Xi_{\mathsf{P}} & \Xi_{\mathsf{A}}^T & \Pi_Z^T \\ \Xi_{\mathsf{A}} & \Xi_{\mathsf{P}} & 0 \\ \Pi_Z & 0 & \beta I_{p+m} \end{bmatrix} > 0, \quad (30)$$

$$\Pi_Z := \begin{bmatrix} \widehat{Q}^{\frac{1}{2}}C_1 & \widehat{Q}^{\frac{1}{2}}C_1 Y \\ \widehat{R}^{\frac{1}{2}}MC_2 & \widehat{R}^{\frac{1}{2}}F \end{bmatrix}.$$

To verify the validity of our method, we introduce the following nominal control problem (**Nop**):

$$\min_{X,Y,F,L,W,\beta} \beta \quad \text{s.t.} \quad (30).$$

The nominal controller without delta-sigma modulation mechanism is given by (33). In the nominal control, we apply the controller (33) to the isolation table with on-off drive input.

Consider the case that h=2ms, T_d=10ms, $\widehat{Q}$=1, $\widehat{R}$ = 5, μ = 0.01 and d=2.5×10^{-4}kg/s. The simulation results are shown in Figs. 11 and 12. The impulse disturbance for the displacement velocity of the isolation table is applied for 0.5s. The nominal controller tends to suppress the residual vibration compared with the case without control. The nominal controller has no delta-sigma modulation mechanism and is designed without considering the quantization effect, so it stands to reason that the stationary error remains. On the other hand, the vertical displacement controlled by the proposed controller quickly converges in the neighborhood of the origin and the residual vibration is suppressed compared with the case without control even if the on-off drive control is applied. We see that our method can achieve several control performances within multiobjective LMI technique.

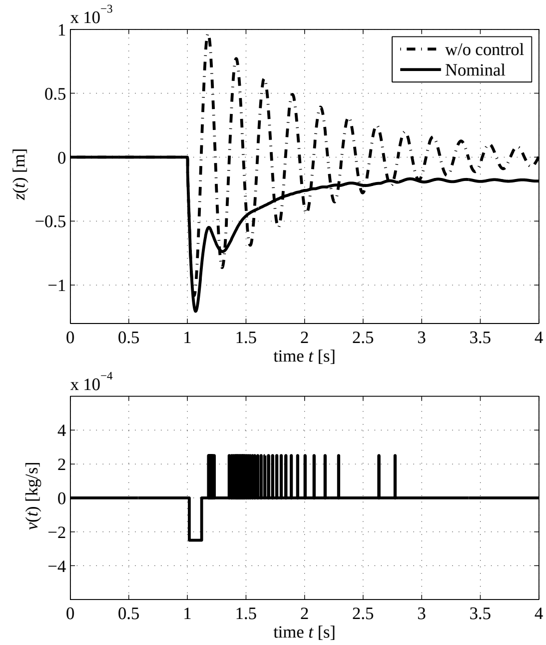

Fig.11: *Simulation results for nominal control*

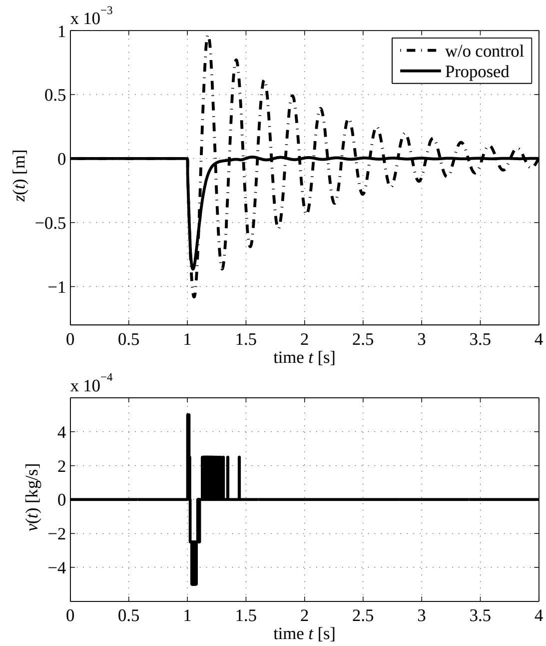

Fig.12: *Simulation results for proposed method*

5.2 Relation to the dynamic quantizer

The goals of the dynamic quantizer synthesis (two-step synthesis) and our simultaneous synthesis are different, although the feedback control systems are both expressed as in Fig. 3. As mentioned in Introduction, the dynamic quantizer synthesis aims to minimize performance degradation before or after the quantizer insert. In contrast, the simultaneous synthesis aims to guarantee a certain performance or several performances within multiobjective control framework. Hence, it makes no sense to compare the two methods to determine which is better. However, it is expected that the simultaneous synthesis is superior to the dynamic quantizer synthesis in some respects. The objective of this subsection is to conduct this sanity check.

Consider the control system with quantized-valued input in Fig. 2 (b) and suppose that the plant P is given by (5) and the nominal controller C is given by

$$\begin{bmatrix} x_c(t+1) \\ u(t) \end{bmatrix} = \begin{bmatrix} A_n & B_{n1} & B_{n2} \\ C_n & D_{n1} & D_{n2} \end{bmatrix} \begin{bmatrix} x_c(t) \\ r(t) \\ y(t) \end{bmatrix} \quad (31)$$

where $x_c \in \mathbb{R}^{n_c}$ and $r \in \mathbb{R}^l$ denote the state and the exogenous signal input, respectively. The dynamic quantizer $v = Q_d(u)$ consists of the static quantizer $q : \mathbb{R}^m \to d\mathbb{N}^m$ with the quantization interval $d \in \mathbb{R}$, i.e.,

$$v = q(u_q + u)$$

and the discrete-time LTI filter $\widehat{Q}$ with the state $x_q \in \mathbb{R}^{n_q}$

$$\begin{bmatrix} x_q(t+1) \\ u_q(t) \end{bmatrix} = \begin{bmatrix} A_q & B_q \\ C_q & 0 \end{bmatrix} \begin{bmatrix} x_q(t) \\ e_q(t) \end{bmatrix}, \quad e_q := v - u. \quad (32)$$

The initial state of $\widehat{Q}$ is $x_q(0) = 0$ for the drift-free. For the quantization error q_e, $e_q = e + u_q$ holds. Then, $Q_2(\mathbf{z})$ of (6) corresponds to

$$u_q = \left(\begin{array}{c|c} A_q + B_q C_q & B_q \\ \hline C_q & 0 \end{array} \right) e.$$

For the system in Fig. 2 (b) with the exogenous signal sequence $R := \{r(0), r(1), ..., \} \in \ell_\infty^p$, $z(t, \hat{x}_0, R)$ denotes the output of z at the t-th time for the initial state $\hat{x}_0 := \left[\, x(0)^T \quad x_c(0)^T \, \right]^T$. Also, for the system in Fig. 2 (a) without the quantizer, $z^*(t, \hat{x}_0, R)$ denotes its output at the t-th time for the initial state $\hat{x}_0$. The cost function for the dynamic quantizer synthesis [7-10] is given by

$$E(\widehat{Q}) := \sup_{(\hat{x}_0, R) \in \mathbb{R}^{n_p + n_c} \times \ell_\infty^p} \|\hat{z}_p(\hat{x}_0, R)\|,$$

$$\|\hat{z}_p(\hat{x}_0, R)\| := \max_i \sup_t |z_i(t, \hat{x}_0, R) - z_i^*(t, \hat{x}_0, R)|.$$

The optimal quantizer minimizing $E(\widehat{Q})$ allows us to approximate the usual system in Fig. 2 (a) by the quantized system in Fig. 2 (b).

The dynamic quantizer synthesis is the following problem (**E**): *For the systems (5) and (31) with the exogenous signal sequence $R \in \ell_\infty^p$, suppose that the quantization interval $d \in \mathbb{R}_+$ is given. Find a dynamic quantizer Q_d (i.e., find parameters (n_q, A_q, B_q, C_q)) minimizing $E(\widehat{Q})$.*

Regarding the solutions to the problem (**E**), please see Appendix B. If the systems (5) and (31) are minimum phase, the optimal filter is given by $\widehat{Q}_{op}$ in (36). Otherwise, the suboptimal filter is given by $\widehat{Q}_{sub}$ in (37). The both filter orders are $n_q = n_p + n_c$, so the obtained quantizers may be (in some cases very) high order. On the other hand, the order of our simultaneous synthesis controller is $n_Q = n_p$ and less than that of the dynamic quantizer.

Consider the active control of pneumatic isolation table in the same fashion. The discretized system including the time delay is $n_p = 9$, so the nominal controller obtained from (**Nop**) is also $n_c = 9$. That is, the filter $\widehat{Q}$ is $n_q = 18$ and the resulting control system order is 36. Also, the discretized model of the isolation table is non-minimum phase, so we have to find a suboptimal filter $\widehat{Q}_{sub}$ from (**Dop**). However, this approach has a computational issue; the numerical precision of LMI solver depends on the size of the decision variables. In contrast, the simultaneous controller in the previous subsection is $n_Q = 9$ and the resulting control system order is 18. This advantage is significant in terms of implementation.

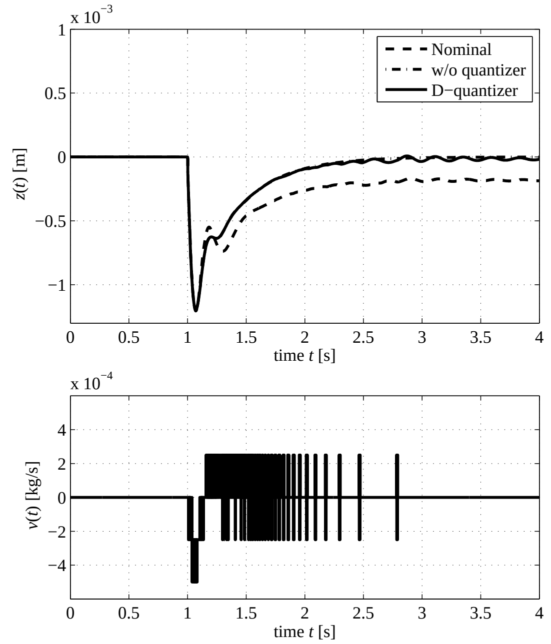

Fig.13: *Simulation results for dynamic quantizer*

The simulation result before or after the dynamic quantizer insert is shown in Fig. 13. The dashed line is for the nominal controller as shown in Fig. 11. The chained line is for the nominal controller without the static quantizer (the usual controlled output of Fig. 2 (a)). The solid line is for the nominal controller with the dynamic quantizer (the controlled output of Fig. 2 (b)). We see that the stationary error is suppressed and the controlled output of the system with the dynamic quantizers approximates that of the usual system. The transient response for the dynamic quantizer synthesis is slow, and this is caused by the particular choice of the nominal controller. Thus, the example suggests that the nominal controller should be designed carefully in the dynamic quantizer synthesis. Our simultaneous synthesis can be viewed as a method for designing a reasonable nominal controller.

6. CONCLUSION

Focusing on the feedback control problems for systems with quantized input, we have proposed the output feedback controller synthesis conditions. The synthesis problem we address is the simultaneous synthesis of the nominal controller and the delta-sigma modulator (where the modulators are called the dynamic quantizers in [8-14]). First, this paper has proposed the controller analysis condition. Second, this paper has proposed the synthesis condition that is recast as a set of matrix inequality condition. Third, an optimal controller for the quantized feedback system has been clarified within the invariant set framework. We also have discussed the controller synthesis conditions which are characterized by the transmission zero property.Finally, we have verified the validity of our proposed method and clarified the following contributions.

- The proposed method naturally extends to multi-objective control.
- The system order for our simultaneous synthesis is less than that of the dynamic quantizer synthesis. This is significant advantage in terms of implementation.
- A numerical example has shown that our simultaneous synthesis allows us to design a reasonable nominal controller and to improve the transient response performance more sharply than the dynamic quantizer.

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APPENDIX

A NOMINAL CONTROLLER DESIGN

Assuming $v = v_Q$ ($Q_2(\mathbf{z})$ does not operate), we introduce the following lemma for the nominal controller $Q_1(\mathbf{z})$.

Lemma 3: For the quantized feedback control system Σ_Q, suppose that $v = v_Q$ ($Q_2(\mathbf{z})$ does not operate) and the performance level $\beta \in \mathbb{R}_+$ is given. There exists a controller $Q_1(\mathbf{z})$ achieving (29) if one of the following equivalent statements holds.

(*i*) There exist a matrix $0 < \mathcal{P} \in \mathbb{R}^{n \times n}$ and a controller $Q_1(\mathbf{z})$ satisfying (28).

(*ii*) There exist matrices $0 < X \in \mathbb{R}^{n_p \times n_p}$, $0 < Y \in \mathbb{R}^{n_p \times n_p}$, $F \in \mathbb{R}^{m \times n_p}$, $L \in \mathbb{R}^{n_p \times m}$, $W \in \mathbb{R}^{n_p \times n_p}$ and $M \in \mathbb{R}^{m \times m}$ satisfying (30).

One such controller parameter ($n_Q = n_p$) is given by

$$\begin{bmatrix} A_Q & B_{Q1} \\ C_Q & D_Q \end{bmatrix} = \begin{bmatrix} Z & XBc \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} W - XAY & L \\ F & M \end{bmatrix} \begin{bmatrix} -Y & 0 \\ C_2Y & I \end{bmatrix}^{-1} \quad (33)$$

where $Z = X - Y^{-1}$.

Proof: We define $V(t) = x_\Sigma(t)^T \mathcal{P} x_\Sigma(t)$. We see that (28) is a sufficient condition of

$$\begin{aligned} & x_\Sigma(t)^T \mathcal{A}^T \mathcal{P} \mathcal{A} x_\Sigma(t) - x_\Sigma(t)^T \mathcal{P} x_\Sigma(t) \\ & \qquad + \tfrac{1}{\beta} x_\Sigma(t)^T \mathcal{C}_3^T \mathcal{C}_3 x_\Sigma(t) \\ & = V(t+1) - V(t) + \tfrac{1}{\beta} x_\Sigma(t)^T \mathcal{C}_3^T \mathcal{C}_3 x_\Sigma(t) < 0. \end{aligned}$$

The above inequality ensures the Lyapunov stability of the feedback control system. Also, summing from $t = 0$ to ∞ and noting the stability property, we have.

$$\begin{aligned} J_{cost} = \sum_{t=0}^{\infty} x_\Sigma(t)^T \mathcal{C}_3^T \mathcal{C}_3 x_\Sigma(t) \\ < \beta(V(0) - V(\infty)) < \beta V(0). \end{aligned}$$

We then see that the above inequality ensures (29). Inequality (28) can be rewritten as

$$\begin{bmatrix} \mathcal{P} & \mathcal{A}^T\mathcal{P} & \mathcal{C}_3^T \\ \mathcal{P}\mathcal{A} & \mathcal{P} & 0 \\ \mathcal{C}_3 & 0 & \beta I \end{bmatrix} > 0. \quad (34)$$

For the full order case $n_Q = n_p$, without loss of generality, $\mathcal{P}$ has the special structure (18). Introducing the change of variables

$$\begin{aligned} \begin{bmatrix} W & L \\ F & M \end{bmatrix} := \begin{bmatrix} XAY & 0 \\ 0 & 0 \end{bmatrix} \\ + \begin{bmatrix} Z & XB \\ 0 & I_m \end{bmatrix} \begin{bmatrix} A_Q & B_{Q1} \\ C_Q & D_Q \end{bmatrix} \begin{bmatrix} -Y & 0 \\ C_2Y & I_m \end{bmatrix}, \end{aligned}$$

the congruence transformation of the condition in (34) and $\mathcal{P} > 0$ by diag (T, T, I) and T yields (30) and

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0 \quad (35)$$

where T is given by (19). (35) is included in (30). ■

B DYNAMIC QUANTIZER

Define the following matrices

$$\widehat{A} := \begin{bmatrix} A + BD_{n2}C_2 & BC_n \\ B_{n2}C_2 & A_n \end{bmatrix}, \quad \widehat{B} := \begin{bmatrix} B \\ 0 \end{bmatrix},$$
$$\widehat{C} = \begin{bmatrix} C_1 & 0 \end{bmatrix}.$$

Assume that (i) the matrix $\widehat{C}\widehat{A}^\tau\widehat{B}$ is non-singular for the smallest integer $\tau \in \{0\} \cup \mathbb{N}_+$ satisfying $\widehat{C}\widehat{A}^\tau\widehat{B} \neq 0$, (ii) the matrix $[\ D_{n2}C_2 \quad C_n\ ]$ is full row rank and (iii) the usual feedback system composed of (5) and (30) is minimum phase. In this case, an optimal solution of the problem (**E**) [7-10] is given by

$$\widehat{Q}_{op} : \begin{cases} n_q = n_p + n_c, & A_q = \widehat{A}, \\ B_q = \widehat{B}, & C_q = -(\widehat{C}\widehat{A}^\tau\widehat{B})^\dagger \widehat{A}^{\tau+1} \end{cases} \quad (36)$$

and its achievable performance is given by

$$E(\widehat{Q}_{op}) = \|\mathrm{abs}(\widehat{C}\widehat{A}^\tau\widehat{B})\| \frac{d}{2}.$$

In the case where the assumptions (i), (ii) and (iii) do not hold, a suboptimal solution is given by the numerical optimization problem [11, 13, 14] (**Dop**):

$$\min_{X>0, Y>0, F, W, U, 1>\alpha>0, \gamma>0} \gamma$$
$$\text{s.t.} \quad \begin{bmatrix} (1-\alpha)\Psi_\mathsf{P} & 0 & \Psi_\mathsf{A}^T \\ 0 & \alpha I_m & \Psi_\mathsf{B}^T \\ \Psi_\mathsf{A} & \Psi_\mathsf{B} & \Psi_\mathsf{P} \end{bmatrix} \geq 0, \quad \begin{bmatrix} \Psi_\mathsf{P} & \Psi_\mathsf{C}^T \\ \Psi_\mathsf{C} & \gamma^2 I_p \end{bmatrix} \geq 0$$

where

$$\Psi_\mathsf{P} := \begin{bmatrix} X & I \\ I & Y \end{bmatrix}, \quad \Psi_\mathsf{A} := \begin{bmatrix} X\widehat{A} & W \\ \widehat{A} & \widehat{A}Y + \widehat{B}F \end{bmatrix},$$
$$\Psi_\mathsf{B} := \begin{bmatrix} U \\ \widehat{B} \end{bmatrix}, \quad \Psi_\mathsf{C} := \begin{bmatrix} \widehat{C} & \widehat{C}Y \end{bmatrix}.$$

A suboptimal dynamic quantizer is given by

$$\widehat{Q}_{sub} : \begin{cases} n_q = n_p + n_c, \quad Z = X - Y^{-1}, \\ A_q = Z^{-1}(X\widehat{A}Y + UF - W)Y^{-1}, \\ B_q = Z^{-1}(U - X\widehat{B}), \quad C_q = -FY^{-1} \end{cases} \quad (37)$$

and its achievable performance is given by

$$E(\widehat{Q}_{sub}) \leq \gamma \frac{\sqrt{m}d}{2}.$$

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